A NOTE ON p-COMPLETION OF SPECTRA

GEOFFROY HOREL

ABSTRACT. We compare two notions of p-completion for a spectrum.

There are two endofunctors of the category of spectra that can legitimately be called completion at the prime p. The first one is the localization at the Moore spectrum $S\mathbb{Z}/p$ and is what is usually called p-completion and the second one is the unit map $X \mapsto \mathrm{Mat}(\widehat{X})$ in an adjunction between spectra and pro-objects in the category of spectra whose homotopy groups are finite p-groups and almost all zero. The goal of this note is to compare these two functors. This will be achieved in Theorem 2.3.

This result should not come as a surprise to experts in the field. However, it seems to be missing from the literature. The main reason is probably that, in order to formulate it precisely, one needs an ∞-categorical notion of pro-categories.

Acknowledgement. I wish to thank Tobias Barthel and Drew Heard for helpful conversations.

1. Notations

This note is written using the language of ∞-categories. All the categorical notion should be understood in the ∞-categorical sense. All the ∞-categories that we consider are stable and we denote by Map their mapping spectrum.

For E a spectrum, we denote by c_pE the localization of E with respect to the homology theory $S\mathbb{Z}/p$. Note that by [Bou79, Theorem 3.1.], this coincides with the $H\mathbb{Z}/p$ localization if E is bounded below. For E a bounded below spectrum with finitely generated homotopy groups, the map $E \to c_pE$ induces the map $E_* \to E_* \otimes \mathbb{Z}_p$ on homotopy groups.

We denote by $\mathbf{Sp}_{p-fin}$ the smallest full stable subcategory of $\mathbf{Sp}$ containing the spectrum $H\mathbb{Z}/p$. This is also the full subcategory of $\mathbf{Sp}$ spanned by the spectra whose homotopy groups are finite p-groups and are almost all 0.

We denote by $\widehat{\mathbf{Sp}}_p$ the category $\mathrm{Pro}(\mathbf{Sp}_{p-fin})$. The inclusion $\mathbf{Sp}_{p-fin} \to \mathbf{Sp}$ induces a limit preserving functor $\mathrm{Mat} : \widehat{\mathbf{Sp}}_p \to \mathbf{Sp}$. This has a left adjoint denoted $X \mapsto \widehat{X}$. We often use the notation $X = \{X_i\}_{i\in I}$ for objects of $\widehat{\mathbf{Sp}}_p$. This means that $X = \lim_i X_i$ in $\widehat{\mathbf{Sp}}_p$ with I cofiltered and X_i in $\mathbf{Sp}_{p-fin}$. Any object of $\widehat{\mathbf{Sp}}_p$ admits a presentation of this form. The functor $\mathrm{Mat}(X)$ is then given by the formula $\mathrm{Mat}(X) = \lim_I X_i$ where the limit is computed in $\mathbf{Sp}$.

We denote by τ_n the n-th Postnikov section endofunctor on $\mathbf{Sp}$. By the universal property of the pro-category, there is a unique endofunctor of $\widehat{\mathbf{Sp}}_p$ that coincides with τ_n on $\mathbf{Sp}_{p-fin}$ and commutes with cofiltered limits. For A a pro-p abelian group, we denote by $\widehat{H}A$ the object of $\widehat{\mathbf{Sp}}_p$ given by applying the Eilenberg-MacLane functor to an inverse system of finite abelian group whose limit is A. Note for instance that $\widehat{H}\mathbb{Z}_p$ lives in $\widehat{\mathbf{Sp}}_p$ while $H\mathbb{Z}_p$ lives in $\mathbf{Sp}$. We obviously have a weak equivalence $H\mathbb{Z}_p \to \mathrm{Mat}(\widehat{H}\mathbb{Z}_p)$.

We denote by $\mathbf{Sp}_p^{ft}$ the full subcategory of $\mathbf{Sp}$ spanned by bounded below spectra whose homotopy groups are finitely generated $\mathbb{Z}_p$-modules. Note that if X is a bounded below spectrum that has finitely generated homotopy groups, then c_pX is in $\mathbf{Sp}_p^{ft}$. Similarly, we denote by $\widehat{\mathbf{Sp}}_p^{ft}$ the full subcategory of $\widehat{\mathbf{Sp}}_p$ spanned by pro-spectra that are bounded below and whose homotopy groups are finitely generated $\mathbb{Z}_p$-modules (given a pro-spectrum $X = \{X_i\}_{i\in I}$, its n-th homotopy group is the pro-abelian group $\{\pi_n(X_i)\}_{i\in I}$).

2. Proofs

Lemma 2.1. *The map $H\mathbb{Z}_p \to \mathrm{Mat}(\widehat{H}\mathbb{Z}_p)$ is adjoint to a weak equivalence $\widehat{H\mathbb{Z}_p} \to \widehat{H}\mathbb{Z}_p$.*

Proof. It suffices to show that for any spectrum F in $\mathbf{Sp}_{p-fin}$, the map

$$\mathrm{Map}_{\widehat{\mathbf{Sp}_p}}(\widehat{H}\mathbb{Z}_p, F) \to \mathrm{Map}(H\mathbb{Z}_p, F)$$

is a weak equivalence. Since both sides of the equation are exact in F, it suffices to do it for $F = H\mathbb{Z}/p$. Hence we are reduced to proving that the map

$$\mathrm{colim}_n H^k(H\mathbb{Z}/p^n, \mathbb{Z}/p) \to H^k(H\mathbb{Z}_p, \mathbb{Z}/p)$$

is an isomorphism for each k. Since $\mathbb{Z}/p$ is a field, cohomology is dual to homology and it suffices to prove that $H_k(H\mathbb{Z}_p, \mathbb{Z}_p)$ is isomorphic to $\{H_k(H\mathbb{Z}/p^n, \mathbb{Z}/p)\}_n$ in the category of pro-abelian groups. In [Lur11, Proposition 3.3.10.], Lurie shows that there is an isomorphism of pro-abelian groups:

$$H_k(\Sigma^{-m}\Sigma^\infty K(\mathbb{Z}_p, m), \mathbb{Z}/p) \cong \{H_k(\Sigma^{-m}\Sigma^\infty K(\mathbb{Z}/p^n, m), \mathbb{Z}/p)\}_n$$

By Freudenthal suspension theorem, for any abelian group A, the map $\Sigma^{-m}\Sigma^\infty K(A,m) \to HA$ is about m-connected. Thus, taking m large enough, Lurie's result gives what we need. $\square$

Proposition 2.2. *Let Y be an object of $\mathbf{Sp}_p^{ft}$. Then the unit map $Y \to \mathrm{Mat}(\widehat{Y})$ is a weak equivalence.*

Proof. Let us call a spectrum Y good if this is the case. The good spectra form a triangulated subcategory of $\mathbf{Sp}$. This subcategory contains $H\mathbb{Z}/p$. According to Lemma 2.1, it also contains $H\mathbb{Z}_p$. Hence, it contains $\tau_n Y$ for any n and any Y in $\mathbf{Sp}_p^{ft}$.

Thus, for Y in $\mathbf{Sp}_p^{ft}$, there is an equivalence $\tau_n Y \to \mathrm{Mat}(\widehat{\tau_n Y})$ for each n. In order to prove that Y is good, it will be enough to prove that the map

$$\mathrm{Mat}(\widehat{Y}) \to \lim_n \mathrm{Mat}(\widehat{\tau_n Y})$$

is a weak equivalence. Since Mat is a right adjoint, it is enough to prove that the obvious map

$$\widehat{Y} \to \lim_n \widehat{\tau_n Y}$$

is a weak equivalence. As in the previous lemma, it is enough to prove that for each k the map

$$\mathrm{colim}_n H^k(\tau_n Y, \mathbb{Z}/p) \to H^k(Y, \mathbb{Z}/p)$$

is an isomorphism which is straightforward. $\square$

We can now prove our main result.

Theorem 2.3. *There is a natural transformation from c_p to $\mathrm{Mat}(\widehat{-})$ that is a weak equivalence when restricted to spectra X such that c_pX is in $\mathbf{Sp}_p^{ft}$. In particular, it is a weak equivalence on spectra that are bounded below and have finitely generated homotopy groups.*

Proof. We first make the observation that for any spectrum X, the obvious map $\widehat{X} \to \widehat{c_pX}$ is a weak equivalence. Indeed, it suffices to prove that for any F in $\mathbf{Sp}_{p-fin}$, the map $X \to c_pX$ induces a weak equivalence

$$\mathrm{Map}(c_pX, F) \to \mathrm{Map}(X, F)$$

but this follows from the fact that F is local with respect to $S\mathbb{Z}/p$.

Thus, there is a natural transformation of endofunctors of $\mathbf{Sp}$:

$$\alpha(X) : c_pX \to \mathrm{Mat}(\widehat{c_pX}) \simeq \mathrm{Mat}(\widehat{X})$$

Proposition 2.2 tells us that $\alpha(X)$ is a weak equivalence whenever $c_p(X)$ is in $\mathbf{Sp}_p^{ft}$ as desired. □

3. The Adams spectral sequence

As an application of Theorem 2.3, we give an alternative construction of the Adams spectral sequence. We denote by H the Eilenberg-MacLane spectrum $H\mathbb{Z}/p$. We denote by $\mathscr{A}$ the ring spectrum $\mathrm{Map}(H,H)$. Note that $\mathscr{A}^* = \pi_{-*}\mathscr{A}$ is the Steenrod algebra. There is a functor $\mathbf{Sp}^{\mathrm{op}} \to \mathbf{Mod}_{\mathscr{A}}$ sending X to $\mathrm{Map}(X,H)$.

Proposition 3.1. *Let X be any spectrum and Y be an object of $\mathbf{Sp}_p^{ft}$. Then, the obvious map*

$$\mathrm{Map}(X,Y) \to \mathrm{Map}_{\mathscr{A}}(\mathrm{Map}(Y,H), \mathrm{Map}(X,H))$$

is a weak equivalence.

Proof. Since H is in $\mathbf{Sp}_{p-fin}$, we have an equivalence $\mathrm{Map}(X,H) \simeq \mathrm{Map}_{\widehat{\mathbf{Sp}}_p}(\widehat{X},H)$ for any spectrum X. By 2.2, the map

$$\mathrm{Map}(X,Y) \to \mathrm{Map}_{\widehat{\mathbf{Sp}}_p}(\widehat{X},\widehat{Y})$$

is an equivalence. Hence, we are reduced to proving that the obvious map

$$\mathrm{Map}_{\widehat{\mathbf{Sp}}}(\widehat{X},\widehat{Y}) \to \mathrm{Map}_{\mathscr{A}}(\mathrm{Map}(\widehat{Y},H), \mathrm{Map}(\widehat{X},H))$$

is an equivalence. We claim more generally that for any object Z in $\widehat{\mathbf{Sp}}_p$, the map

$$\mathrm{Map}_{\widehat{\mathbf{Sp}}}(\widehat{X},Z) \to \mathrm{Map}_{\mathscr{A}}(\mathrm{Map}(Z,H), \mathrm{Map}(\widehat{X},H))$$

is an equivalence. Indeed, since both sides are limit preserving in the variable Z, it suffices to prove it for $Z = H\mathbb{Z}/p$ which is tautological. □

According to [EKMM97, Theorem IV.4.1.], we get a conditionally convergent spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathscr{A}^*}(H^*(Y), H^*(X)) \implies \pi_{s+t}\mathrm{Map}(X,Y)$$

for Y a spectrum in $\mathbf{Sp}_p^{ft}$.

References

[Bou79] A. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[Lur11] J. Lurie. Derived algebraic geometry XIII. *available at http://www.math.harvard.edu/ lurie/*, 2011.

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Deutschland
E-mail address: geoffroy.horel@gmail.com