MIXED HODGE STRUCTURES AND FORMALITY OF SYMMETRIC MONOIDAL FUNCTORS

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Abstract. We use mixed Hodge theory to show that the functor of singular chains with rational coefficients is formal as a lax symmetric monoidal functor, when restricted to complex schemes whose weight filtration in cohomology satisfies a certain purity property. This has direct applications to the formality of operads or, more generally, of algebraic structures encoded by a colored operad. We also prove a dual statement, with applications to formality in the context of rational homotopy theory. In the general case of complex schemes with non-pure weight filtration, we relate the singular chains functor to a functor defined via the first term of the weight spectral sequence.

1. Introduction

There is a long tradition of using Hodge theory as a tool for proving formality results. The first instance of this idea can be found in [DGMS75] where the authors prove that compact Kähler manifolds are formal (i.e. the commutative differential graded algebra of differential forms is quasi-isomorphic to its cohomology). In the introduction of that paper, the authors explain that their intuition came from the theory of étale cohomology and the fact that the degree n étale cohomology group of a smooth projective variety over a finite field is pure of weight n. This purity is what morally prevents the existence of non-trivial Massey products. In the setting of complex algebraic geometry, Deligne introduced in [Del71, Del74] a filtration on the rational cohomology of every complex algebraic variety X, called the *weight filtration*, with the property that each of the successive quotients of this filtration behaves as the cohomology of a smooth projective variety, in the sense that it has a Hodge-type decomposition. Deligne's mixed Hodge theory was subsequently promoted to the rational homotopy of complex algebraic varieties (see [Mor78], [Hai87], [NA87]). This can then be used to make the intuition of the introduction of [DGMS75] precise. In [Dup16] and [CC17], it is shown that purity of the weight filtration in cohomology implies formality, in the sense of rational homotopy, of the underlying topological space. However, the treatment of the theory in these references lacks functoriality and is restricted to smooth varieties in the first paper and to projective varieties in the second.

In another direction, in the paper [GNPR05], the authors elaborate on the method of [DGMS75] and prove that operads (as well as cyclic operads, modular operads, etc.) internal to the category of compact Kähler manifolds are formal. Their strategy is to introduce the functor of de Rham currents which is a functor from compact Kähler manifolds to chain complexes that is symmetric monoidal and quasi-isomorphic to the singular chain functor as a lax symmetric monoidal functor. Then they show that this functor is formal as a lax symmetric monoidal functor. Recall that, if $\mathcal{C}$ is a symmetric monoidal category and $\mathcal{A}$ is an

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abelian symmetric monoidal category, a lax symmetric monoidal functor $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ is said to be formal if it is weakly equivalent to $H_* \circ F$ in the category of lax symmetric monoidal functors. It is then straightforward to see that such functors send operads in $\mathcal{C}$ to formal operads in $\mathrm{Ch}_*(\mathcal{A})$. The functoriality also immediately gives us that a map of operads in $\mathcal{C}$ is sent to a formal map of operads or that an operad with an action of a group G is sent to a formal operad with a G-action. Of course, there is nothing specific about operads in these statements and they would be equally true for monoids, cyclic operads, modular operads, or more generally any algebraic structure that can be encoded by a colored operad.

The purpose of this paper is to push this idea of formality of symmetric monoidal functors from complex algebraic varieties in several directions in order to prove the most general possible theorem of the form "purity implies formality". Before explaining our results more precisely, we need to introduce a bit of terminology.

Let X be a complex algebraic variety. Deligne's weight filtration on the rational n-th cohomology vector space of X is bounded by

$$0 = W_{-1}H^n(X, \mathbb{Q}) \subseteq W_0H^n(X, \mathbb{Q}) \subseteq \cdots \subseteq W_{2n}H^n(X, \mathbb{Q}) = H^n(X, \mathbb{Q}).$$

If X is smooth then $W_{n-1}H^n(X, \mathbb{Q}) = 0$, while if X is projective $W_nH^n(X, \mathbb{Q}) = H^n(X, \mathbb{Q})$. In particular, if X is a smooth and projective then we have

$$0 = W_{n-1}H^n(X, \mathbb{Q}) \subseteq W_nH^n(X, \mathbb{Q}) = H^n(X, \mathbb{Q}).$$

In this case, the weight filtration on $H^n(X, \mathbb{Q})$ is said to be *pure of weight* n. More generally, for α a rational number and X a complex algebraic variety, we say that the weight filtration on $H^*(X, \mathbb{Q})$ is *α-pure* if, for all $n \geq 0$, we have

$$Gr_p^W H^n(X, \mathbb{Q}) := \frac{W_pH^n(X, \mathbb{Q})}{W_{p-1}H^n(X, \mathbb{Q})} = 0 \text{ for all } p \neq \alpha n.$$

The bounds on the weight filtration tell us that this makes sense only when $0 \leq \alpha \leq 2$. Note as well that if we write $\alpha = a/b$ with $(a, b) = 1$, α-purity implies that the cohomology is concentrated in degrees that are divisible by b, and that $H^{bn}(X, \mathbb{Q})$ is pure of weight an.

Aside from smooth projective varieties, some well-known examples of varieties with 1-pure weight filtration are: projective V-manifolds, projective varieties whose underlying topological space is a $\mathbb{Q}$-homology manifold and the moduli spaces $\mathcal{M}_{Dol}$ and $\mathcal{M}_{dR}$ appearing in the non-abelian Hodge correspondence. Complements of hyperplane arrangements and complements of toric arrangements as well as the moduli spaces $\mathcal{M}_{0,n}$ of smooth projective curves of genus 0 with n marked points make examples with 2-pure weight filtration. As we shall see in Section 8, complements of codimension d subspace arrangements are examples of smooth schemes whose weight filtration in cohomology is $2d/(2d-1)$-pure.

Our main result is Theorem 7.3. We show that, for a non-zero rational number α, the singular chains functor

$$S_*(-, \mathbb{Q}) : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to complex schemes whose weight filtration in cohomology is α-pure. Here $\mathrm{Sch}_{\mathbb{C}}$ denotes the category of complex schemes, that are reduced, separated and of finite type. This generalizes the main result of [GNPR05] on the formality of $S_*(X, \mathbb{Q})$ for any operad X on smooth projective varieties, to the case of operads in possibly singular and/or non-compact varieties with pure weight filtration in cohomology.

As direct applications of the above result, we prove formality of the operad of singular chains of some operads in complex schemes, such as the noncommutative analogue of the (framed) little 2-discs operad introduced in [DSV15] and the monoid of self-maps of the complex projective line studied by Cazanave in [Caz12] (see Theorems 7.4 and 7.7). We also reinterpret in the language of mixed Hodge theory the proofs of the formality of the little disks operad and Getzler's gravity operad appearing in [Pet14] and [DH17]. These last two results do not fit directly in our framework, since the little disks operad and the gravity operad do not quite come from operads in algebraic varieties. However, the action of the Grothendieck-Teichmüller group provides a bridge to mixed Hodge theory.

In Theorem 8.1 we prove a dual statement of our main result, showing that Sullivan's functor of piece-wise linear forms

$$\mathcal{A}^*_{PL} : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to schemes whose weight filtration in cohomology is α-pure, where α is a non-zero rational number.

This gives functorial formality in the sense of rational homotopy for such schemes, generalizing both "purity implies formality" statements appearing in [Dup16] for smooth varieties and in [CC17] for singular projective varieties. Our generalization is threefold: we allow rational weights, obtain functoriality and we study possibly singular and open varieties simultaneously.

Theorems 7.3 and 8.1 deal with situations in which the weight filtration is pure. In the general context with mixed weights, it was shown by Morgan [Mor78] for smooth schemes and in [CG14] for possibly singular schemes, that the first term of the multiplicative weight spectral sequence carries all the rational homotopy information of the scheme. In Theorem 7.8 we provide the analogous statement for the lax symmetric monoidal functor of singular chains. A dual statement for Sullivan's functor of piece-wise linear forms is proven in Theorem 8.7, enhancing the results of [Mor78] and [CG14] with functoriality.

We now explain the structure of this paper. The first four sections are purely algebraic. In Section 2 we collect the main properties of formal lax symmetric monoidal functors that we use. In particular, in Theorem 2.3 we recall a recent theorem of rigidification due to Hinich that says that, over a field of characteristic zero, formality of functors can be checked at the level of ∞-functors. We also introduce the notion of α-purity for complexes of bigraded objects in a symmetric monoidal abelian category and show that, when restricted to α-pure complexes, the functor defined by forgetting the degree is formal.

The connection of this result with mixed Hodge structures is done in Section 3, where we prove a symmetric monoidal version of Deligne's weak splitting of mixed Hodge structures over $\mathbb{C}$. Such splitting is a key tool towards formality. In Section 4 we study lax monoidal functors to vector spaces over a field of characteristic zero equipped with a compatible filtration. We show, in Theorem 4.3, that the existence of a lax monoidal splitting for such functors is independent of the field. As a consequence, we obtain splittings for the weight filtration over $\mathbb{Q}$. This result enables us to bypass the theory of descent of formality for operads of [GNPR05], which assumes the existence of minimal models. Putting the above results together we are able to show that the forgetful functor

$$\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}}) \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

induced by sending a rational mixed Hodge structure to its underlying vector space, is formal when restricted to those complexes whose mixed Hodge structure in homology is α-pure.

In order to obtain a symmetric monoidal functor from the category of complex schemes to an algebraic category encoding mixed Hodge structures, we have to consider more flexible objects than complexes of mixed Hodge structures. This is the content of Section 5, where we study the category $\mathrm{MHC}_{\Bbbk}$ of mixed Hodge complexes. In Theorem 5.4 we explain a promotion of Beilinson's equivalence of categories $D^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathrm{ho}(\mathrm{MHC}_{\Bbbk})$ between the derived category of mixed Hodge structures and the homotopy category of mixed Hodge complexes, to an equivalence of symmetric monoidal ∞-categories (see also [Dre15], [BNT15]).

The geometric character of this paper comes in Section 6, where we construct a symmetric monoidal functor from complex schemes to mixed Hodge complexes. This is done in two steps. First, for smooth schemes, we dualize Navarro's construction [NA87] of functorial mixed Hodge complexes to obtain a lax monoidal ∞-functor

$$\mathcal{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

such that its composite with the forgetful functor $\mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ is naturally weakly equivalent to $S_*(-,\mathbb{Q})$ as a lax symmetric monoidal ∞-functor (see Theorem 6.4). Note that in order to obtain monoidality, we move to the world of ∞-categories, denoted in boldface letters. In the second step, we extend this functor from smooth, to singular schemes, by standard descent arguments.

The main results of this paper are stated and proven in Section 7, where we also explain several applications to operad formality. Lastly, Section 8 contains applications to the rational homotopy theory of complex schemes.

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Notations. As a rule, we use boldface letters to denote ∞-categories and normal letters to denote 1-categories. For $\mathcal{C}$ a 1-category, we denote by $\mathbf{N}(\mathcal{C})$ its nerve seen as an ∞-category. If $\mathcal{C}$ is a relative category we also use $\mathbf{N}(\mathcal{C})$ for the ∞-categorical localization of $\mathcal{C}$ at its weak equivalences.

For $\mathcal{A}$ an additive category, we will denote by $\mathrm{Ch}_*^?(\mathcal{A})$ the category of (homologically graded) chain complexes in $\mathcal{A}$, where ? denotes the boundedness condition: nothing for unbounded, b for bounded below and above and ≥ 0 (resp. ≤ 0 for non-negatively (resp. non-positively) graded complexes. We denote by $\mathbf{Ch}_*^?(\mathcal{A})$ the ∞-category obtained from $\mathrm{Ch}_*^?(\mathcal{A})$ by inverting the quasi-isomorphisms.

2. Formal symmetric monoidal functors

Let $(\mathcal{A}, \otimes, \mathbf{1})$ be an abelian symmetric monoidal category. The homology functor H_* : $\mathrm{Ch}_*(\mathcal{A}) \longrightarrow \prod_{n\in\mathbb{Z}} \mathcal{A}$ is a lax symmetric monoidal functor, via the usual Künneth morphism. In the cases that will interest us, all the objects of $\mathcal{A}$ will be flat and the homology functor is in fact strong symmetric monoidal.

We recall the following definition from [GNPR05].

Definition 2.1. Let $\mathcal{C}$ be a symmetric monoidal category and $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ a lax symmetric monoidal functor. Then F is said to be a *formal lax symmetric monoidal* functor if F and $H_* \circ F$ are weakly equivalent in the category of lax symmetric monoidal functors: there is a string of monoidal natural transformations of lax symmetric monoidal functors

$$F \xleftarrow{\Phi_1} F_1 \longrightarrow \cdots \longleftarrow F_n \xrightarrow{\Phi_n} H_* \circ F$$

such that for every object X of $\mathcal{C}$, the morphisms $\Phi_i(X)$ are quasi-isomorphisms.

Definition 2.2. Let $\mathcal{C}$ be a symmetric monoidal category and $F : \mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathcal{A})$ a lax symmetric monoidal functor (in the ∞-categorical sense). We say that F is a *formal* *lax symmetric monoidal ∞-functor* if F and $H_* \circ F$ are weakly equivalent as lax monoidal functors from $\mathbf{N}(\mathcal{C})$ to $\mathbf{Ch}_*(\mathcal{A})$.

Clearly a formal lax symmetric monoidal functor $\mathcal{C} \to \mathrm{Ch}_*(\mathcal{A})$ induces a formal lax symmetric monoidal ∞-functor $\mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathcal{A})$. The following theorem and its corollary give a partial converse.

Theorem 2.3 (Hinich)**.** *Let $\Bbbk$ be a field of characteristic 0. Let $\mathcal{C}$ be a small symmetric monoidal category. Let F and G be two lax symmetric monoidal functors $\mathcal{C} \to \mathrm{Ch}_*(\Bbbk)$. If F and G are weakly equivalent as lax symmetric monoidal ∞-functors $\mathbf{N}(\mathcal{C}) \longrightarrow \mathbf{Ch}_*(\Bbbk)$, then F and G are weakly equivalent as lax symmetric monoidal functors.*

Proof. This theorem is true more generally if $\mathcal{C}$ is a colored operad. Indeed recall that any symmetric monoidal category has an underlying colored operad whose category of algebras is equivalent to the category of lax monoidal functors out of the original category.

Now since we are working in characteristic zero, the operad underlying $\mathcal{C}$ is homotopically sound (following the terminology of [Hin15]). Therefore, [Hin15, Theorem 4.1.1] gives us an equivalence of ∞-categories

$$\mathbf{N}(\mathrm{Alg}_{\mathcal{C}}(\mathrm{Ch}_*(\Bbbk)) \xrightarrow{\sim} \mathbf{Alg}_{\mathcal{C}}(\mathbf{Ch}_*(\Bbbk))$$

where we denote by $\mathrm{Alg}_{\mathcal{C}}$ (resp. $\mathbf{Alg}_{\mathcal{C}}$) the category of lax monoidal functors (resp. the ∞-category of lax monoidal functors) out of $\mathcal{C}$. Now, the two functors F and G are two objects in the source of the above map that become weakly equivalent in the target. Hence, they are already equivalent in the source, which is precisely saying that they are connected by a zig-zag of weak equivalences of lax monoidal functors. □

Corollary 2.4. *Let $\Bbbk$ be a field of characteristic 0. Let $\mathcal{C}$ be a small symmetric monoidal category. Let $F : \mathcal{C} \to \mathrm{Ch}_*(\Bbbk)$ be a lax symmetric monoidal functor. If F is formal as lax symmetric monoidal ∞-functor $\mathbf{N}(\mathcal{C}) \longrightarrow \mathbf{Ch}_*(\Bbbk)$, then F is formal as lax symmetric monoidal functor.*

Proof. It suffices to apply Theorem 2.3 to F and $H_* \circ F$. □

The following proposition whose proof is trivial is the reason we are interested in formal lax monoidal functors.

Proposition 2.5 ([GNPR05], Proposition 2.5.5)**.** *If $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ is a formal lax symmetric monoidal functor then F sends operads in $\mathcal{C}$ to formal operads in $\mathrm{Ch}_*(\mathcal{A})$.*

In rational homotopy, there is a criterion of formality in terms of weight decompositions which proves to be useful in certain situations (see for example [BMSS98] and [BD78]). We next provide an analogous criterion in the setting of symmetric monoidal functors.

Denote by $gr\mathcal{A}$ the category of graded objects of $\mathcal{A}$. It inherits a symmetric monoidal structure from that of $\mathcal{A}$, with the tensor product defined by

$$(A \otimes B)^n := \bigoplus_p A^p \otimes B^{p-n}.$$

The unit in $gr\mathcal{A}$ is given by $\mathbf{1}$ concentrated in degree zero. The functor $U : gr\mathcal{A} \longrightarrow \mathcal{A}$ obtained by forgetting the degree is symmetric monoidal. The category of graded complexes $\mathrm{Ch}_*(gr\mathcal{A})$ inherits a symmetric monoidal structure via a graded Künneth morphism.

Definition 2.6. Given a rational number α, denote by $\mathrm{Ch}_*(gr\mathcal{A})^{\alpha\text{-pure}}$ the full subcategory of $\mathrm{Ch}_*(gr\mathcal{A})$ given by those graded complexes $A = \bigoplus A_n^p$ with *α-pure homology*:

$$H_n(A)^p = 0 \text{ for all } p \neq \alpha n.$$

Note that if $\alpha = a/b$, with a and b coprime, then the above condition implies that $H_*(A)$ is concentrated in degrees that are divisible by b, and in degree kb, it is pure of weight ka:

$$H_{kb}(A)^p = 0 \text{ for all } p \neq ka.$$

Proposition 2.7. *Let $\mathcal{A}$ be an abelian category and α a non-zero rational number. The functor $U : \mathrm{Ch}_*(gr\mathcal{A})^{\alpha\text{-}pure} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ defined by forgetting the degree is formal as a lax symmetric monoidal functor.*

Proof. We will define a functor $\tau : \mathrm{Ch}_*(gr\mathcal{A}) \longrightarrow \mathrm{Ch}_*(gr\mathcal{A})$ together with natural transformations

$$\Phi : U \circ \tau \Rightarrow U \text{ and } \Psi : U \circ \tau \Rightarrow H \circ U$$

giving rise to weak equivalences when restricted to chain complexes with α-pure homology.

Consider the truncation functor $\tau : \mathrm{Ch}_*(gr\mathcal{A}) \longrightarrow \mathrm{Ch}_*(gr\mathcal{A})$ defined by sending a graded chain complex $A = \bigoplus A_n^p$ to the graded complex given by:

$$(\tau A)_n^p := \begin{cases} A_n^p & n > \lceil p/\alpha \rceil \\ \mathrm{Ker}(d : A_n^p \to A_{n-1}^p) & n = \lceil p/\alpha \rceil \\ 0 & n < \lceil p/\alpha \rceil \end{cases},$$

where $\lceil p/\alpha \rceil$ denotes the smallest integer greater than or equal to p/α. Note that for each p, $\tau(A)_*^p$ is the chain complex given by the canonical truncation of A_*^p at $\lceil p/\alpha \rceil$, which satisfies

$$H_n(\tau(A)_*^p) \cong H_n(A_*^p) \text{ for all } n \geq \lceil p/\alpha \rceil.$$

To prove that τ is a lax symmetric monoidal functor it suffices to see that

$$\tau(A)_n^p \otimes \tau(B)_m^q \subseteq \tau(A \otimes B)_{n+m}^{p+q}$$

for all $A, B \in \mathrm{Ch}_*(gr\mathcal{A})$. It suffices to consider three cases:

(1) If $n > \lceil p/\alpha \rceil$ and $m \geq \lceil q/\alpha \rceil$ then $n + m > \lceil p/\alpha \rceil + \lceil q/\alpha \rceil \geq \lceil (p+q)/\alpha \rceil$. Therefore we have $\tau(A \otimes B)_{n+m}^{p+q} = (A \otimes B)_{n+m}^{p+q}$ and the above inclusion is trivially satisfied.

(2) If $n = \lceil p/\alpha \rceil$ and $m = \lceil q/\alpha \rceil$ then $n + m = \lceil p/\alpha \rceil + \lceil q/\alpha \rceil \geq \lceil (p+q)/\alpha \rceil$. Now, if $n + m > \lceil (p+q)/\alpha \rceil$ then again we have $\tau(A \otimes B)_{n+m}^{p+q} = (A \otimes B)_{n+m}^{p+q}$. If $n + m = \lceil (p+q)/\alpha \rceil$ then the above inclusion reads

$$\mathrm{Ker}(d : A_n^p \to A_{n-1}^p) \otimes \mathrm{Ker}(d : B_m^q \to B_{m-1}^q) \subseteq \mathrm{Ker}(d : (A \otimes B)_{n+m}^{p+q} \to (A \otimes B)_{n+m-1}^{p+q}).$$

This is verified by the Leibniz rule.

(3) Lastly, if $n < \lceil p/\alpha \rceil$ then $\tau(A)_n^p = 0$ and there is nothing to verify.

The projection $\mathrm{Ker}(d : A_n^p \to A_{n-1}^p) \twoheadrightarrow H_n(A)^p$ defines a morphism $\tau A \to H(A)$ by

$$(\tau A)_n^p \mapsto \begin{cases} 0 & n \neq \lceil p/\alpha \rceil \\ H_n(A)^p & n = \lceil p/\alpha \rceil \end{cases}.$$

This gives a monoidal natural transformation $\Psi : U \circ \tau \Rightarrow H \circ U = U \circ H$. Likewise, the inclusion $\tau A \hookrightarrow A$ defines a monoidal natural transformation $\Phi : U \circ \tau \Rightarrow U$.

Let A be a complex of $\mathrm{Ch}_*(gr\mathcal{A})^{\alpha\text{-pure}}$. Then both morphisms

$$\Psi(A) : \tau \circ U(A) \to H \circ U(A) \text{ and } \Phi(A) : U \circ \tau(A) \to U(A)$$

are clearly quasi-isomorphisms. □

For graded chain complexes whose homology is pure up to a certain degree, we obtain a result of partial formality as follows.

Definition 2.8. Let $q \geq 0$ be an integer. A morphism of chain complexes $f : A \to B$ is called *q-quasi-isomorphism* if the induced morphism in homology $H_i(f) : H_i(A) \to H_i(B)$ is an isomorphism for all $i \leq q$ and an epimorphism for $i = q + 1$.

Definition 2.9. Let $q \geq 0$ be an integer. A functor $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ is a *q-formal lax symmetric monoidal* functor if the maps $\Phi_i(X)$ in Definition 2.1 are q-quasi-isomorphism for all $1 \leq i \leq n$.

Proposition 2.10. *Let $\mathcal{A}$ be an abelian category. Given a non-zero rational number α and an integer $q \geq 0$, denote by $\mathrm{Ch}_*(gr\mathcal{A})_q^{\alpha\text{-}pure}$ the full subcategory of $\mathrm{Ch}_*(gr\mathcal{A})$ given by those graded complexes $A = \bigoplus A_n^p$ whose homology in degrees $\leq q+1$ is α-pure: for all $n \leq q+1$,*

$$H_n(A)^p = 0 \text{ for all } p \neq \alpha n.$$

Then the functor $U : \mathrm{Ch}_*(gr\mathcal{A})_q^{\alpha\text{-}pure} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ defined by forgetting the degree is q-formal.

Proof. The proof is parallel to that of Proposition 2.7 by noting that, if $H_n(A)$ is α-pure for $n \leq q+1$, then the morphisms

$$\Psi(A) : \tau \circ U(A) \to H \circ U(A) \text{ and } \Phi(A) : U \circ \tau(A) \to U(A)$$

are q-quasi-isomorphisms. □

3. Mixed Hodge structures

Denote by $\mathcal{FA}$ the category of filtered objects of an abelian symmetric monoidal category $(\mathcal{A}, \otimes, \mathbf{1})$. All filtrations will be assumed to be of finite length and exhaustive. With the tensor product

$$W_p(A \otimes B) := \sum_{i+j=p} \mathrm{Im}(W_i A \otimes W_j B \longrightarrow A \otimes B),$$

and the unit given by $\mathbf{1}$ concentrated in weight zero, $\mathcal{FA}$ is a symmetric monoidal category. The functor $U^{fil} : gr\mathcal{A} \longrightarrow \mathcal{FA}$ defined by $A = \bigoplus A^p \mapsto W_m A := \bigoplus_{q \leq m} A^q$ is symmetric monoidal. The category of filtered complexes $\mathrm{Ch}_*(\mathcal{FA})$ inherits a symmetric monoidal structure via a filtered Künneth morphism and we have a symmetric monoidal functor

$$U^{fil} : \mathrm{Ch}_*(gr\mathcal{A}) \longrightarrow \mathrm{Ch}_*(\mathcal{FA}).$$

Let $\Bbbk \subset \mathbb{R}$ be a subfield of the real numbers.

Definition 3.1. A *mixed Hodge structure* on a finite dimensional $\Bbbk$-vector space V is given by a filtration W of V, called the *weight filtration*, together with a filtration F on $V_{\mathbb{C}} := V \otimes \mathbb{C}$, called the *Hodge filtration*, such that for all $m \geq 0$, each $\Bbbk$-module $Gr_m^W V := W_m V / W_{m-1} V$ carries a pure Hodge structure of weight m given by the filtration induced by F on $Gr_m^W V \otimes \mathbb{C}$, that is, there is a direct sum decomposition

$$Gr_W^m V \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q} \text{ where } V^{p,q} = F^p(Gr_m^W V \otimes \mathbb{C}) \cap \overline{F}^q(Gr_m^W V \otimes \mathbb{C}) = \overline{V}^{q,p}.$$

Morphisms of mixed Hodge structures are given by morphisms $f : V \to V'$ of $\Bbbk$-modules compatible with filtrations: $f(W_m V) \subset W_m V'$ and $f(F^p V_{\mathbb{C}}) \subset F^p V'_{\mathbb{C}}$. Denote by $\mathrm{MHS}_{\Bbbk}$ the category of mixed Hodge structures over $\Bbbk$. It is an abelian category by [Del71, Theorem 2.3.5].

Remark 3.2. Given mixed Hodge structures V and V', then $V \otimes V'$ is a mixed Hodge structure with the filtered tensor product. This makes $\mathrm{MHS}_{\Bbbk}$ into a symmetric monoidal category. Also, $\mathsf{Hom}(V, V')$ is a mixed Hodge structure, with the weight filtration given by

$$W_p\mathsf{Hom}(V, V') := \{f : V \to V'; f(W_q V) \subset W_{q+p} V', \forall q\}$$

and the Hodge filtration defined in the same way. In particular, the dual of a mixed Hodge structure is again a mixed Hodge structure.

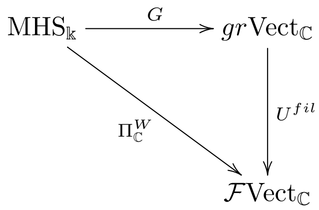
$\Bbbk \subset \mathbb{K}$ be a field extension. The functors

$$\Pi_{\mathbb{K}} : \mathrm{MHS}_{\Bbbk} \longrightarrow \mathrm{Vect}_{\mathbb{K}} \text{ and } \Pi^W_{\mathbb{K}} : \mathrm{MHS}_{\Bbbk} \longrightarrow \mathcal{F}\mathrm{Vect}_{\mathbb{K}}$$

defined by sending a mixed Hodge structure (V, W, F) to $V_{\mathbb{K}} := V_{\Bbbk} \otimes \mathbb{K}$ and $(V_{\mathbb{K}}, W)$ respectively, are symmetric monoidal functors.

Deligne introduced a global decomposition of $V_{\mathbb{C}} := V \otimes \mathbb{C}$ into subspaces $I^{p,q}$, for any mixed Hodge structure (V, W, F) which generalizes the decomposition of pure Hodge structures of a given weight. In this case, one has a congruence $I^{p,q} \equiv \overline{I}^{q,p}$ modulo W_{p+q-2}. We study this decomposition in the context of symmetric monoidal functors.

Lemma 3.3 (Deligne's splitting)**.** *The functor $\Pi^W_{\mathbb{C}}$ admits a factorization*



into symmetric monoidal functors. In particular, there is an isomorphism of functors

$$U^{fil} \circ gr \circ \Pi^W_{\mathbb{C}} \cong \Pi^W_{\mathbb{C}},$$

where $gr : \mathcal{F}\mathrm{Vect}_{\mathbb{C}} \longrightarrow gr\mathrm{Vect}_{\mathbb{C}}$ is the graded functor given by $gr(V_{\mathbb{C}}, W)^p = Gr^W_p V_{\mathbb{C}}$.

Proof. Let (V, W, F) be a mixed Hodge structure. By [Del71, 1.2.11] (see also [GS75, Lemma 1.12]), there is a direct sum decomposition $V_{\mathbb{C}} = \bigoplus I^{p,q}(V)$ where

$$I^{p,q}(V) = (F^p W_{p+q} V_{\mathbb{C}}) \cap \left(\overline{F}^q W_{p+q} V_{\mathbb{C}} + \sum_{i>0} \overline{F}^{q-i} W_{p+q-1-i} V_{\mathbb{C}} \right).$$

This decomposition is functorial for morphisms of mixed Hodge structures and satisfies

$$W_m V_{\mathbb{C}} = \bigoplus_{p+q \leq m} I^{p,q}(V).$$

Define G by letting $G(V, W, F)^n := \bigoplus_{p+q=n} I^{p,q}(V)$ for any mixed Hodge structure. Since $f(I^{p,q}(V)) \subset I^{p,q}(V')$ for every morphism $f : (V, W, F) \to (V', W, F)$ of mixed Hodge structures, G is symmetric monoidal. The functor $U^{fil} : gr\mathrm{Vect} \longrightarrow \mathcal{F}\mathrm{Vect}$ is the symmetric monoidal functor given by

$$\bigoplus_n V^n \mapsto (V, W), \text{ with } W_m V := \bigoplus_{n \leq m} V^n.$$

Therefore we have $U^{fil} \circ G = \Pi^W_{\mathbb{C}}$. In order to prove the isomorphism $U^{fil} \circ gr \circ \Pi^W_{\mathbb{C}} \cong \Pi^W_{\mathbb{C}}$ it suffices to note that there is an isomorphism of functors $gr \circ U^{fil} \cong \mathrm{Id}$. □

4. Descent of splittings of lax monoidal functors

In this section, we study lax monoidal functors to vector spaces over a field of characteristic zero $\Bbbk$ equipped with a compatible filtration. More precisely, we are interested in lax monoidal maps $\mathcal{C} \longrightarrow \mathcal{F}\mathrm{Vect}_{\Bbbk}$. Our goal is to prove that the existence of a lax monoidal splitting for such a functor, i.e. of a lift of this map to $\mathcal{C} \longrightarrow gr\mathrm{Vect}_{\Bbbk}$, does not depend on the field $\Bbbk$. Our proof follows similar arguments to those appearing in [CG14, Section 2.4], see also [GNPR05] and [Sul77]. A main advantage of our approach with respect to these references is that, in proving descent at the level of functors, we avoid the use of minimal models (and thus restrictions to, for instance, operads with trivial arity 0).

It will be a bit more convenient to study a more general situation where $\mathcal{C}$ is allowed to be a colored operad instead of a symmetric monoidal category. Indeed recall that any symmetric monoidal category can be seen as an operad whose colors are the objects of $\mathcal{C}$ and where a multimorphism from $(c_1, \ldots, c_n)$ to d is just a morphism in $\mathcal{C}$ from $c_1 \otimes \ldots \otimes c_n$ to d. Then, given another symmetric monoidal category $\mathcal{D}$, there is an equivalence of categories between the category of lax monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and the category of $\mathcal{C}$-algebras in the symmetric monoidal category $\mathcal{D}$.

We fix (V, W) a map of colored operads $\mathcal{C} \longrightarrow \mathcal{F}\mathrm{Vect}_{\Bbbk}$ such that for each color c of $\mathcal{C}$, the vector space $V(c)$ is finite dimensional. We denote by $\mathsf{Aut}_W(V)$ the set of its automorphisms in the category of $\mathcal{C}$-algebras in $\mathcal{F}\mathrm{Vect}_{\Bbbk}$ and by $\mathsf{Aut}(Gr^W V)$ the set of automorphisms of $Gr^W V$ in the category of $\mathcal{C}$-algebras in $gr\mathrm{Vect}_{\Bbbk}$. We have a morphism $gr : \mathsf{Aut}_W(V) \to \mathsf{Aut}(Gr^W V)$.

Let $\Bbbk \to R$ be a commutative $\Bbbk$-algebra. The correspondence

$$R \mapsto \mathbf{Aut}_W(V)(R) := \mathsf{Aut}_W(V \otimes_{\Bbbk} R)$$

defines a functor $\mathbf{Aut}_W(V) : \mathrm{Alg}_{\Bbbk} \longrightarrow \mathrm{Gps}$ from the category $\mathrm{Alg}_{\Bbbk}$ of commutative $\Bbbk$-algebras, to the category Gps of groups. Clearly, we have $\mathbf{Aut}_W(V)(\Bbbk) = \mathsf{Aut}_W(V)$. We define in a similar fashion a functor $\mathbf{Aut}(Gr^W V)$ from $\mathrm{Alg}_{\Bbbk}$ to Gps.

We recall the following properties:

Proposition 4.1. *Let (V, W) be as above.*

1. $\mathsf{Aut}_W(V)$ *is a pro-algebraic matrix group over* $\Bbbk$.
2. $\mathbf{Aut}_W(V)$ *is a pro-algebraic affine group scheme over* $\Bbbk$ *represented by* $\mathsf{Aut}_W(V)$.
3. *The grading morphism gr defines a morphism* $\mathbf{gr} : \mathbf{Aut}_W(V) \to \mathbf{Aut}(Gr^W V)$ *of pro-algebraic affine group schemes.*
4. *The kernel* $\mathbf{N} := \mathrm{Ker}\left(\mathbf{gr} : \mathbf{Aut}_W(V) \to \mathbf{Aut}(Gr^W V)\right)$ *is a pro-unipotent algebraic affine group scheme over* $\Bbbk$.

Proof. We can write $\mathcal{C}$ as a filtered colimit of suboperads with finitely many objects. Then the category of algebras is just the limit of the category of algebras for each of these suboperads. Hence, in order to prove this proposition, it is enough to show it when $\mathcal{C}$ has finitely many objects and when we remove the prefix pro everywhere.

Let N be such that the vector space $\oplus_{c \in \mathcal{C}} V(c)$ can be linearly embedded in $\Bbbk^N$. Then $\mathsf{Aut}_W(V)$ is the closed subgroup of $\mathbf{GL}_N(\Bbbk)$ defined by the polynomial equations that express the data of a lax monoidal natural filtration preserving automorphism. Thus $\mathsf{Aut}_W(V)$ is an algebraic matrix group. Moreover, $\mathbf{Aut}_W(V)$ is obviously the algebraic affine group scheme represented by $\mathsf{Aut}_W(V)$. Hence (1) and (2) are satisfied.

For every commutative $\Bbbk$-algebra R, the map

$$\mathbf{Aut}_W(V)(R) = \mathsf{Aut}_W(V \otimes_{\Bbbk} R) \longrightarrow \mathsf{Aut}(Gr^W V \otimes_{\Bbbk} R) = \mathbf{Aut}(Gr^W V)(R)$$

is a morphism of groups which is natural in R. Thus (3) follows.

Since by (2) both groups $\mathbf{Aut}_W(V)$ and $\mathbf{Aut}(grV)$ are algebraic and $\Bbbk$ has characteristic zero, the kernel $\mathbf{N}$ is represented by an algebraic matrix group defined over $\Bbbk$ (see [Bor91, Corollary 15.4]). Therefore to prove (4) it suffices to verify that all elements in $\mathbf{N}(\Bbbk)$ are unipotent. We see that it is enough to show that for any f in $\mathbf{N}(\Bbbk)$ and any $c \in \mathcal{C}$, the restriction $f(c)$ to $V(c)$ is unipotent. Consider the Jordan decomposition $f = f_s \cdot f_u$ into semi-simple and unipotent parts. We want to show that $f_s(c) = 1$ for all c. By [Bor91, Theorem 4.4] we have $f_s(c), f_u(c) \in \mathbf{Aut}_W(V(c))(\Bbbk)$. Since $grf(c) = 1$ and an algebraic group morphism preserves semi-simple and unipotent parts, we deduce that

$gr(f_s(c)) = gr(f_u(c)) = 1$. Let $V_1(c) = \mathrm{Ker}(f_s(c) - I)$ and decompose $V(c)$ into $f_s(c)$-invariant subspaces $V = V_1(c) \oplus V'(c)$. Therefore we have $grV(c) = grV_1(c) \oplus grV'(c)$. Since $grV(c)$ contains nothing but the eigenspaces of eigenvalue 1, we have $grV'(c) = 0$, and so $V'(c) = 0$. Therefore $f_s(c) = 1$ and $f(c)$ is unipotent. □

Lemma 4.2. *Let (V, W) be as above. The following are equivalent:*

(1) The pair (V, W) admits a lax monoidal splitting: $W_pV \cong \bigoplus_{q\leq p} Gr^W_q V$.
(2) The morphism $gr : \mathsf{Aut}_W(V) \to \mathsf{Aut}(Gr^W V)$ is surjective.
(3) There exists $\alpha \in \Bbbk^*$ which is not a root of unity together with an automorphism $\Phi \in \mathsf{Aut}_W(V)$ such that $gr(\Phi) = \psi_\alpha$ is the grading automorphism of $Gr^W V$ associated with α, defined by

$$\psi_\alpha(a) = \alpha^p a, \text{ for } a \in Gr^W_p V.$$

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial. We show that (3) implies (1). Let $\Phi \in \mathsf{Aut}_W(V)$ be such that $gr\Phi = \psi_\alpha$. Consider a Jordan decomposition $\Phi = \Phi_s \cdot \Phi_u$. Note that the Jordan decomposition exists even for a pro-algebraic affine group scheme, it suffices to do it levelwise. Moreover, we have the property that for each object c of $\mathcal{C}$, the restrictions $(\Phi_u(c), \Phi_v(c))$ to $V(c)$ form a Jordan decomposition of $\Phi(c) = \Phi_{|V(c)}$. By [Bor91, Theorem 4.4] we have that $\Phi_s(c), \Phi_u(c) \in \mathsf{Aut}_W(V(c))$ and there is a decomposition of the form $V(c) = V'(c) \oplus U(c)$, where

$$V'(c) = \bigoplus V_p(c) \text{ with } V_p(c) := \mathrm{Ker}(\Phi_s(c) - \alpha^p I)$$

and $U(c)$ is the complementary subspace corresponding to the remaining factors of the characteristic polynomial of $\Phi_s(c)$. As in the proof of Proposition 4.1 (4) one concludes that $U(c) = 0$.

In order to show that $W_pV = \bigoplus_{i\leq p} V_p$ it suffices to prove it objectwise. Let c be an object of $\mathcal{C}$. For $x \in V_p(c)$, let q be the smallest integer such that $x \in W_qV(c)$. Then x defines a class $x + W_{q+1}V(c) \in grV(c)$, and

$$\psi_\alpha(x + W_{q-1}V(c)) = \alpha^q x + W_{q-1}V(c) = \Phi(x) + W_{q-1}V(c) = \alpha^p x + W_{q-1}V(c).$$

Then $(\alpha^q - \alpha^p)x \in W_{q-1}V(c)$. Since $x \notin W_{q-1}V(c)$ we have $q = p$, hence $x \in W_pV$. □

We may now state and prove the main theorem of this section.

Theorem 4.3. *Let (V, W) be a map of colored operad $\mathcal{C} \longrightarrow \mathcal{F}\mathrm{Vect}_\Bbbk$ such that for each color c of $\mathcal{C}$, the vector space $V(c)$ is finite dimensional. Let $\Bbbk \subset \mathbb{K}$ be a field extension. Then V admits a lax monoidal splitting if and only if $V_\mathbb{K} := V \otimes_\Bbbk \mathbb{K} : \mathcal{C} \longrightarrow \mathrm{Vect}_\mathbb{K}$ admits a lax monoidal splitting.*

Proof. We may assume without loss of generality that $\mathbb{K}$ is algebraically closed. If $V_\mathbb{K}$ admits a splitting, the map

$$\mathbf{Aut}_W(V)(\mathbb{K}) \longrightarrow \mathbf{Aut}(gr(V))(\mathbb{K})$$

is surjective by Lemma 4.2. From [Wat79, Section 18.1] there is an exact sequence of groups

$$1 \longrightarrow \mathbf{N}(\Bbbk) \longrightarrow \mathbf{Aut}_W(V)(\Bbbk) \longrightarrow \mathbf{Aut}(grV)(\Bbbk) \longrightarrow H^1(\mathbb{K}/\Bbbk, \mathbf{N}) \longrightarrow \dots$$

where $\mathbf{N}$ is pro-unipotent by Proposition 4.1. Since $\Bbbk$ has characteristic zero the group $H^1(\mathbb{K}/\Bbbk, \mathbf{N})$ is trivial (see [Wat79, Example 18.2.e]). This gives the exact sequence

$$1 \to \mathbf{N}(\Bbbk) \longrightarrow \mathsf{Aut}_W(V) \longrightarrow \mathsf{Aut}(gr(V)) \longrightarrow 1.$$

Hence V admits a splitting by Lemma 4.2. □

From this theorem we deduce that Deligne's splitting holds over $\mathbb{Q}$. We have the following lemma.

Lemma 4.4 (Deligne's splitting over $\mathbb{Q}$)**.** *The forgetful functor* $\Pi^W_{\mathbb{Q}} : \mathrm{MHS}_{\mathbb{Q}} \longrightarrow \mathcal{F}\mathrm{Vect}_{\mathbb{Q}}$ *given by* $(V, W, F) \mapsto (V, W)$ *admits a factorization*

$$\begin{array}{ccc} \mathrm{MHS}_{\mathbb{Q}} & \xrightarrow{G} & gr\mathrm{Vect}_{\mathbb{Q}} \\ & {\scriptstyle \Pi^W_{\mathbb{Q}}} \searrow & \downarrow {\scriptstyle U^{fil}} \\ & & \mathcal{F}\mathrm{Vect}_{\mathbb{Q}} \end{array}$$

into lax symmetric monoidal functors. In particular, there is an isomorphism of functors

$$U^{fil} \circ gr \circ \Pi^W_{\mathbb{Q}} \cong \Pi^W_{\mathbb{Q}},$$

where $gr : \mathcal{F}\mathrm{Vect}_{\mathbb{Q}} \longrightarrow gr\mathrm{Vect}_{\mathbb{Q}}$ *is the graded functor given by* $gr(V_{\mathbb{Q}}, W)^p = Gr^W_p V_{\mathbb{Q}}$.

Proof. We apply Theorem 4.3 to the lax monoidal functor $\Pi^W_{\mathbb{Q}}$ using the fact that $\Pi^W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ admits a splitting by Lemma 3.3. □

Remark 4.5. We want to emphasize that Theorem 4.3 does not say that the splitting of the previous lemma recovers the splitting of Lemma 3.3 after tensoring with $\mathbb{C}$. In fact, it can probably be shown that such a splitting cannot exist. Nevertheless, the existence of Deligne's splitting over $\mathbb{C}$ abstractly forces the existence of a similar splitting over $\mathbb{Q}$ which is all this Lemma is saying. Note as well that these are not splittings of mixed Hodge structures, but only of the weight filtration. They are also referred to as *weak splittings* of mixed Hodge structures (see for example [PS08, Section 3.1]). As is well-known, mixed Hodge structures do not split in general.

The above splitting over $\mathbb{Q}$ yields the following "purity implies formality" statement in the abstract setting of functors defined from the category of complexes of mixed Hodge structures. Given a rational number α, denote by $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-}pure}$ the full subcategory of $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$ of complexes with pure weight α homology: an object (K, W, F) in $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-}pure}$ is such that $Gr^p_W H_n(K) = 0$ for all $p \neq \alpha n$.

Corollary 4.6. *The restriction of the functor* $\Pi_{\mathbb{Q}} : \mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}}) \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$ *to the category* $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})^{\alpha\text{-}pure}$ *is formal for any non-zero rational number* α.

Proof. This follows from Proposition 2.7 together with Lemma 4.4. □

5. Mixed Hodge complexes

We next recall the notion of mixed Hodge complex introduced by Deligne in [Del74] in its chain complex version (with homological degree). Let $\Bbbk \subset \mathbb{R}$ be a subfield of the real numbers.

Definition 5.1. A *mixed Hodge complex over* $\Bbbk$ is given by a filtered bounded chain complex $(K_{\Bbbk}, W)$ over $\Bbbk$, a bifiltered chain complex $(K_{\mathbb{C}}, W, F)$ over $\mathbb{C}$, together with a finite string of filtered quasi-isomorphisms of filtered complexes of $\mathbb{C}$-modules

$$(K_{\Bbbk}, W) \otimes \mathbb{C} \xrightarrow{\alpha_1} (K_1, W) \xleftarrow{\alpha_2} \cdots \xrightarrow{\alpha_{l-1}} (K_{l-1}, W) \xrightarrow{\alpha_l} (K_{\mathbb{C}}, W).$$

We call l the *length* of the mixed Hodge complex. The following axioms must be satisfied:

(MH_0) The homology $H_*(K_{\Bbbk})$ is of finite type.

(MH_1) The differential of $Gr_W^p K_{\mathbb{C}}$ is strictly compatible with F.
(MH_2) The filtration on $H_n(Gr_W^p K_{\mathbb{C}})$ induced by F makes $H_n(Gr_W^p K_{\Bbbk})$ into a pure Hodge structure of weight $p+n$.

Such a mixed Hodge complex will be denoted by $\mathcal{K} = \{(K_{\Bbbk}, W), (K_{\mathbb{C}}, W, F)\}$, omitting the data of the comparison morphisms α_i.

Axiom (MH_2) implies that for all $n \geq 0$ the triple $(H_n(K_{\Bbbk}), \mathrm{Dec}W, F)$ is a mixed Hodge structure over $\Bbbk$, where $\mathrm{Dec}W$ denotes Deligne's décalage of the weight filtration (see [Del71, Definition 1.3.3]).

Morphisms of mixed Hodge complexes are given by levelwise bifiltered morphisms of complexes making the corresponding diagrams commute. Denote by $\mathrm{MHC}_{\Bbbk}$ the category of mixed Hodge complexes of a certain fixed length, which we omit in the notation. The tensor product of mixed Hodge complexes is again a mixed Hodge complex (see [PS08, Lemma 3.20]). This makes $\mathrm{MHC}_{\Bbbk}$ into a symmetric monoidal category, with a filtered variant of the Künneth formula.

Definition 5.2. A morphism $f : K \to L$ in $\mathrm{MHC}_{\Bbbk}$ is said to be a *weak equivalence* if $H_*(f_{\Bbbk})$ is an isomorphism of $\Bbbk$-vector spaces.

Since the category of mixed Hodge structures is abelian, the homology of every complex of mixed Hodge structures is a graded mixed Hodge structure. We have a functor

$$\mathcal{T} : \mathrm{Ch}_*^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathrm{MHC}_{\Bbbk}$$

given on objects by $(K, W, F) \mapsto \{(K, TW), (K \otimes \mathbb{C}, TW, F)\}$, where TW is the shifted filtration $(TW)^p K_n := W^{p+n} K_n$. The comparison morphisms α_i are the identity. Also, $\mathcal{T}$ is the identity on morphisms. This functor clearly preserves weak equivalences.

Lemma 5.3. *The shift functor* $\mathcal{T} : \mathrm{Ch}_*^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathrm{MHC}_{\Bbbk}$ *is symmetric monoidal.*

Proof. It suffices to note that given filtered complexes (K, W) and (L, W), we have:

$$T(W \otimes W)_p (K \otimes L)^n = (TW \otimes TW)_p (K \otimes L)^n. \qquad \square$$

Beilinson [Beĭ86] gave an equivalence of categories between the derived category of mixed Hodge structures and the homotopy category of a shifted version of mixed Hodge complexes. We will require a finer version of Beilinson's equivalence, in terms of symmetric monoidal ∞-categories. Denote by $\mathbf{MHC}_{\Bbbk}$ the ∞-category obtained by inverting weak equivalences of mixed Hodge complexes, omitting the length in the notation. As explained in [Dre15, Theorem 2.7.], this object is canonically a symmetric monoidal stable ∞-category. Note that in loc. cit., mixed Hodge complexes have fixed length 2 and are polarized. The results of [Dre15] as well as Beilinson's equivalence, are equally valid for the category of mixed Hodge complexes of an arbitrary fixed length.

Theorem 5.4. *The shift functor induces an equivalence* $\mathbf{Ch}_*^b(\mathrm{MHS}_{\Bbbk}) \longrightarrow \mathbf{MHC}_{\Bbbk}$ *of symmetric monoidal ∞-categories.*

Proof. A proof in the polarizable setting appears in [Dre15]. Also, in [BNT15], a similar statement is proven for a shifted version of mixed Hodge complexes. We sketch a proof in our setting.

We first observe as in Lemma 2.6 of [BNT15] that both ∞-categories are stable and that the shift functor is exact. The stability of $\mathbf{MHC}_{\Bbbk}$ follows from the observation that this ∞-category is the Verdier quotient at the acyclic complexes of the ∞-category of mixed Hodge complexes with the homotopy equivalences inverted. This last ∞-category underlies a dg-category that can easily be seen to be stable. The stability of $\mathbf{Ch}_*^b(\mathrm{MHS}_{\Bbbk})$ follows in a similar way. Since a complex of mixed Hodge structures is acyclic if and only if the underlying complex of $\mathbb{k}$-modules is acyclic, and $\mathcal{T}$ is the identity on the underlying complexes of $\mathbb{k}$-modules, it follows that $\mathcal{T}$ is exact. Therefore, in order to prove that $\mathcal{T}$ is an equivalence of ∞-categories, it suffices to show that it induces an equivalence of homotopy categories

$$D^b(\mathrm{MHS}_{\mathbb{k}}) \longrightarrow \mathrm{ho}(\mathrm{MHC}_{\mathbb{k}}).$$

In [Beĭ86, Lemma 3.11], it is proven that the shift functor $\mathcal{T} : \mathrm{Ch}^b_*(\mathrm{MHS}^p_{\mathbb{k}}) \longrightarrow \mathrm{MHC}^p_{\mathbb{k}}$ induces an equivalence at the level of homotopy categories. Here the superindex p indicates that the mixed Hodge objects are polarized. But in fact the result remains true if we remove the polarization (see also [CG16, Theorem 4.10] for a proof of this last fact). The fact that $\mathcal{T}$ can be given the structure of a symmetric monoidal ∞-functor follows from the work of Drew in [Dre15]. □

6. Logarithmic de Rham currents

The goal of this section is to construct a symmetric monoidal functor from schemes over $\mathbb{C}$ to mixed Hodge complexes which computes the correct mixed Hodge structure after passing to homology. The construction for smooth schemes is relatively straightforward. It suffices to take a functorial mixed Hodge complex model for the cochains as constructed for instance in [NA87] and dualize it. The monoidality of that functor is slightly tricky as one has to move to the world of ∞-categories for it to exist. Once one has constructed this functor for smooth schemes, it can be extended to more general schemes by standard descent arguments.

We denote by $\mathrm{Sch}_{\mathbb{C}}$ the category of complex schemes that are reduced, separated and of finite type and we denote by $\mathrm{Sm}_{\mathbb{C}}$ the subcategory of smooth schemes. Both of these categories are essentially small (i.e. there is a set of isomorphisms classes of objects) and symmetric monoidal under the cartesian product.

We will make use of the following very simple observation.

Proposition 6.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with finite products seen as symmetric monoidal categories with respect to the product. Then any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a preferred oplax monoidal structure.*

Proof. We need to construct comparison morphisms $F(c\times c') \longrightarrow F(c)\times F(c')$. By definition of the product, there is a unique such functor whose composition with the first projection is the map $F(c\times c') \longrightarrow F(c)$ induced by the first projection $c\times c' \longrightarrow c$ and whose composition with the second projection is the map $F(c \times c') \longrightarrow F(c')$ induced by the second projection $c \times c' \longrightarrow c'$. Similarly, one has a unique map $F(*) \longrightarrow *$. One checks easily that these two maps give F the structure of an oplax monoidal functor. □

6.1. For smooth schemes.

In this section, we construct a lax monoidal functor

$$\mathcal{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

such that its composition with the forgetful functor $\mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ is naturally weakly equivalent to $S_*(-,\mathbb{Q})$ as a lax symmetric monoidal functor.

Let X be a smooth projective complex scheme and $j : U \hookrightarrow X$ an open subscheme such that $D := X - U$ is a normal crossings divisor. Denote by $\mathcal{A}^*_X$ the analytic de Rham complex of the underlying real analytic variety of X and let $\mathcal{A}^*_X(\log D)$ denote the subsheaf of $j_*\mathcal{A}^*_U$ of logarithmic forms in D. This sheaf may be naturally endowed with weight and Hodge filtrations W and F (see 8.6 of [NA87]). Furthermore, Proposition 8.4 of loc. cit. gives a string of quasi-isomorphisms of sheaves of filtered cdga's:

$$(\mathbf{R}_{\mathrm{TW}} j_* \underline{\mathbb{Q}}_U, \tau) \otimes \mathbb{C} \xrightarrow{\sim} (\mathbf{R}_{\mathrm{TW}} j_* \mathcal{A}^*_U, \tau) \xleftarrow{\sim} (\mathcal{A}^*_X(\log D), \tau) \xrightarrow{\sim} (\mathcal{A}^*_X(\log D), W),$$

where τ is the canonical filtration.

In this diagram,

$$\mathbf{R}_{\mathrm{TW}} j_* : \mathrm{Sh}(U, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})) \longrightarrow \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q}))$$

is the functor defined by

$$\mathbf{R}_{\mathrm{TW}} j_* := \mathbf{s}_{\mathrm{TW}} \circ j_* \circ G^+$$

where

$$G^\bullet : \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})) \longrightarrow \Delta\mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q}))$$

is the Godement canonical cosimplicial resolution functor and

$$\mathbf{s}_{\mathrm{TW}} : \Delta\mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})) \longrightarrow \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q}))$$

is the Thom-Whitney simple functor introduced by Navarro in Section 2 of loc. cit. Both functors are symmetric monoidal and hence $\mathbf{R}_{\mathrm{TW}} j_*$ is a symmetric monoidal functor (see [RR16, Section 3.2]). The above string of quasi-isomorphisms gives a commutative algebra object in (cohomological) mixed Hodge complexes after taking global sections. Specifically, the composition

$$\mathbf{R}_{\mathrm{TW}}\Gamma(X, -) := \mathbf{s}_{\mathrm{TW}} \circ \Gamma(X, -) \circ G^+$$

gives a derived global sections functor

$$\mathbf{R}_{\mathrm{TW}}\Gamma(X, -) : \mathrm{Sh}(X, \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})) \longrightarrow \mathrm{Ch}_*^{\leq 0}(\mathbb{Q})$$

which again is symmetric monoidal. There is also a filtered version of this functor defined via the filtered Thom-Whitney simple (see Section 6 of [NA87]). Theorem 8.15 of loc. cit. asserts that by applying the (bi)filtered versions of $\mathbf{R}_{\mathrm{TW}}\Gamma(X, -)$ to each of the pieces of the above string of quasi-isomorphisms, one obtains a commutative algebra object in mixed Hodge complexes $\mathcal{H}dg(X, U)$ whose cohomology gives Deligne's mixed Hodge structure on $H^*(U, \mathbb{Q})$ and such that

$$\mathcal{H}dg(X, U)_{\mathbb{Q}} = \mathbf{R}_{\mathrm{TW}}\Gamma(X, \mathbf{R}_{\mathrm{TW}} j_* \underline{\mathbb{Q}}_U)$$

is naturally quasi-isomorphic to $S^*(U, \mathbb{C})$ (as a cochain complex). This construction is functorial for morphisms of pairs $f : (X, U) \to (X', U')$. The definition of $\mathcal{H}dg(f)$ follows as in the additive setting (see [Hub95, Lemma 6.1.2] for details), by replacing the classical additive total simple functor with the Thom-Whitney simple functor.

Now in order to get rid of the dependence on the compactification, we define for U a smooth scheme over $\mathbb{C}$, a mixed Hodge complex $\mathcal{D}^*(U)$ by the formula

$$\mathcal{D}^*(U) := \mathrm{colim}_{(X,U)}\, \mathcal{H}dg(X, U)$$

where the colimit is taken over the category of pairs (X, U) where X is smooth and proper scheme containing U as an open subscheme, and $X - U$ is a normal crossing divisor. By theorems of Hironaka and Nagata, the category of such pairs is a non-empty filtered category. Note that we have to be slightly careful here as the category of mixed Hodge complexes does not have all filtered colimits. However, we can form this colimit in the category of pairs $(K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F)$ having the structure required in Definition 5.1 but not necessarily satisfying the axioms MH_0, MH_1 and MH_2. Since taking filtered colimit is an exact functor, we deduce from the classical isomorphism between sheaf cohomology and singular cohomology that there is a quasi-isomorphism

$$\mathcal{D}^*(U)_{\mathbb{Q}} \to S^*(U, \mathbb{Q})$$

This shows that the cohomology of $\mathcal{D}^*(U)$ is of finite type and hence, that $\mathcal{D}^*(U)$ satisfies axiom MH_0. The other axioms are similarly easily seen to be satisfied. Moreover, filtered colimits preserve commutative algebra structures, therefore the functor $\mathcal{D}^*$ is a functor from $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$ to commutative algebras in $\mathrm{MHC}_{\mathbb{Q}}$.

Since the coproduct in commutative algebras is the tensor product, we deduce from the dual of Proposition 6.1 that $\mathcal{D}^*$ is canonically a lax symmetric monoidal functor from $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$ to $\mathrm{MHC}_{\mathbb{Q}}$. But since the comparison map

$$\mathcal{D}^*(U)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{D}^*(V)_{\mathbb{Q}} \longrightarrow \mathcal{D}^*(U \times V)_{\mathbb{Q}}$$

is a quasi-isomorphism, this functor extends to a symmetric monoidal ∞-functor

$$\mathcal{D}^* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

Remark 6.2. A similar construction for real mixed Hodge complexes is done in [BT15, Section 3.1]. There is also a similar construction in [Dre15] that includes polarizations.

Now, the category $\mathrm{MHC}_{\mathbb{Q}}$ is equipped with a duality functor. It sends a mixed Hodge complex $\{(K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, W, F)\}$ to the linear duals $\{(K_{\mathbb{Q}}^{\vee}, W^{\vee}), (K_{\mathbb{C}}^{\vee}, W^{\vee}, F^{\vee})\}$ where the dual of a filtered complex is defined as in 3.2. One checks easily that this dual object satisfies the axioms of a mixed Hodge complex. Moreover, the duality functor $\mathrm{MHC}_{\mathbb{Q}}^{\mathrm{op}} \longrightarrow \mathrm{MHC}_{\mathbb{Q}}$ is lax monoidal and the canonical map

$$K^{\vee} \otimes L^{\vee} \longrightarrow (K \otimes L)^{\vee}$$

is a weak equivalence. It follows that the duality functor induces a symmetric monoidal ∞-functor

$$\mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

Composing it with $\mathcal{D}^*$, we get a symmetric monoidal ∞-functor

$$\mathcal{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

Remark 6.3. One should note that $\mathcal{D}^*$ comes from a lax symmetric monoidal functor from $\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}$ to $\mathrm{MHC}_{\mathbb{Q}}$. On the other hand, $\mathcal{D}_*$ is induced by a strict functor which is neither lax nor oplax. Its monoidal structure only exists at the ∞-categorical level.

To conclude this construction, it remains to compare the functor $\mathcal{D}_*(-)_{\mathbb{Q}}$ with the singular chains functor. These two functors are naturally quasi-isomorphic as shown in [NA87] but we will need that they are quasi-isomorphic as symmetric monoidal ∞-functors. We denote by $S_*(-, R)$ the singular chain complex functor from the category of topological spaces to the category of chain complexes over a commutative ring R. The functor $S_*(-, R)$ is lax monoidal. Moreover, the natural map

$$S_*(X, R) \otimes S_*(Y, R) \to S_*(X \times Y, R)$$

is a quasi-isomorphism. This implies that $S_*(-, R)$ induces a symmetric monoidal ∞-functor from the category of topological spaces to the ∞-category $\mathbf{Ch}_*(R)$ of chain complexes over R. We still use the symbol $S_*(-, R)$ to denote this ∞-functor.

Theorem 6.4. *The functors $\mathcal{D}_*(-)_{\mathbb{Q}}$ and $S_*(-, \mathbb{Q})$ are weakly equivalent as symmetric monoidal ∞-functors from $\mathbf{N}(\mathrm{Sm}_{\mathbb{C}})$ to $\mathbf{Ch}_*(\mathbb{Q})$.*

Proof. We introduce the category Man of smooth manifolds. We consider the ∞-category $\mathbf{PSh}(\mathrm{Man})$ of presheaves of spaces on the ∞-category $\mathbf{N}(\mathrm{Man})$. This is a symmetric monoidal ∞-category under the product. We can consider the reflective subcategory $\mathbf{T}$ spanned by presheaves $\mathcal{G}$ satisfying the following two conditions:

(1) Given a hypercover $U_\bullet \to M$ of a manifold M, the induced map

$$\mathcal{G}(M) \to \lim_{\Delta} \mathcal{G}(U_\bullet)$$

is an equivalence.

(2) For any manifold M, the map $\mathcal{G}(M) \to \mathcal{G}(M\times\mathbb{R})$ induced by the projection $M\times\mathbb{R} \to M$ is an equivalence.

The presheaves satisfying these conditions are stable under product, hence the ∞-category $\mathbf{T}$ inherits the structure of a symmetric monoidal locally presentable ∞-category. It has a universal property that we now describe.

Given another symmetric monoidal locally presentable ∞-category $\mathbf{D}$, we denote by $\mathrm{Fun}^{L,\otimes}(\mathbf{T},\mathbf{D})$ the ∞-category of colimit preserving symmetric monoidal functors $\mathbf{T}\to\mathbf{D}$. Then, we claim that the restriction map

$$\mathrm{Fun}^{L,\otimes}(\mathbf{T},\mathbf{D}) \to \mathrm{Fun}^{\otimes}(\mathbf{N}\mathrm{Man},\mathbf{D})$$

is fully faithful and that its essential image is the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{Man},\mathbf{D})$ spanned by the functors F that satisfy the following two properties:

(1) Given a hypercover $U_\bullet \to M$ of a manifold M, the map

$$\mathrm{colim}_{\Delta^{\mathrm{op}}} F(U_\bullet) \to F(M)$$

is an equivalence.

(2) For any manifold M, the map $F(M\times\mathbb{R}) \to F(M)$ induced by the projection $M\times\mathbb{R}\to M$ is an equivalence.

This statement can be deduced from the theory of localizations of symmetric monoidal ∞-categories (see [Hin16, Section 3]).

In particular, there exists an essentially unique symmetric monoidal and colimit preserving functor from $\mathbf{T}$ to $\mathbf{S}$ (the ∞-category of spaces) that is determined by the fact that it sends a manifold M to the simplicial set $\mathrm{Sing}(M)$. This functor is an equivalence of ∞-categories. This is a floklore result. A proof of a model category version of this fact can be found in [Dug01, Proposition 8.3.].

The ∞-category $\mathbf{S}$ is the unit of the symmetric monoidal ∞-category of presentable ∞-categories. It follows that it has a commutative algebra structure (which corresponds to the symmetric monoidal structure coming from the cartesian product) and that it is the initial symmetric monoidal presentable ∞-category. Since $\mathbf{T}$ is equivalent to $\mathbf{S}$ as a symmetric monoidal presentable ∞-category, we deduce that, up to equivalence, there is a unique functor $\mathbf{T}\longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ that is symmetric monoidal and colimit preserving. But, using the universal property of $\mathbf{T}$, we easily see that $S_*(-,\mathbb{Q})$ and $\mathcal{D}_*(-)_{\mathbb{Q}}$ can be extended to symmetric monoidal and colimits preserving functors from $\mathbf{T}$ to $\mathbf{Ch}_*(\mathbb{Q})$. It follows that they must be equivalent. □

6.2. **For schemes.** In this subsection, we extend the construction of the previous subsection to the category of schemes.

We have the site $(\mathrm{Sch}_{\mathbb{C}})_{pro}$ of schemes over $\mathbb{C}$ with the proper topology and the site $(\mathrm{Sm}_{\mathbb{C}})_{pro}$ which is the restriction of this site to the category of smooth schemes (see [Bla16, Section 3.5] for the definition of the proper topology).

Proposition 6.5 (Blanc). *Let $\mathbf{C}$ be a symmetric monoidal presentable ∞-category. We denote by $\mathrm{Fun}^{\otimes}_{pro,}(\mathrm{Sch}_{\mathbb{C}},\mathbf{C})$ the ∞-category of symmetric monoidal functors from $\mathrm{Sch}_{\mathbb{C}}$ to $\mathbf{C}$ whose underlying functor satisfies descent with respect to proper hypercovers. Similarly, we denote by $\mathrm{Fun}^{\otimes}_{pro}(\mathrm{Sm}_{\mathbb{C}},\mathbf{C})$ the ∞-category of symmetric monoidal functors from $\mathrm{Sm}_{\mathbb{C}}$ to $\mathbf{C}$ whose underlying functor satisfies descent with respect to proper hypercovers. The restriction functor*

$$\mathrm{Fun}^{\otimes}_{pro}(\mathrm{Sch}_{\mathbb{C}},\mathbf{C}) \longrightarrow \mathrm{Fun}^{\otimes}_{pro}(\mathrm{Sm}_{\mathbb{C}},\mathbf{C})$$

is an equivalence.

Proof. We have the categories $\mathrm{Fun}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$ and $\mathrm{Fun}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$ of presheaves of simplicial sets over $\mathrm{Sch}_{\mathbb{C}}$ and $\mathrm{Sm}_{\mathbb{C}}$ respectively. These categories are related by an adjunction

$$\pi^* : \mathrm{Fun}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) \leftrightarrows \mathrm{Fun}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) : \pi_*$$

where the right adjoint is just the restriction. Both sides if this adjunction have a symmetric monoidal structure by taking objectwise product. The functor π^* is obviously symmetric monoidal. We can equip both sides with the local model structure. We obtain a Quillen adjunction

$$\pi^* : \mathrm{Fun}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) \leftrightarrows \mathrm{Fun}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet}) : \pi_*$$

between symmetric monoidal model categories. In which the right adjoint is a symmetric monoidal functor. In [Bla16, Proposition 3.22], it is proved that this is a Quillen equivalence. The local model structure on the category $\mathrm{Fun}(\mathrm{Sm}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$ presents the ∞-topos of hypercomplete sheaves over the proper site on $\mathrm{Sm}_{\mathbb{C}}$ and similarly for the local model structure on $\mathrm{Fun}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}}, \mathrm{sSet})$. Therefore, this Quillen equivalence implies that these two ∞-topoi are equivalent. Moreover, as in the proof of 6.4, these topoi, seen as symmetric monoidal presentable ∞-categories under the cartesian product, represent the functor $\mathbf{C} \mapsto \mathrm{Fun}^{\otimes}_{pro}(\mathrm{Sm}_{\mathbb{C}}, \mathbf{C})$ (resp. $\mathbf{C} \mapsto \mathrm{Fun}^{\otimes}_{pro}(\mathrm{Sm}_{\mathbb{C}}, \mathbf{C})$). The result immediately follows. □

Theorem 6.6. *Up to weak equivalences, there is a unique symmetric monoidal functor*

$$\mathcal{D}_* : \mathbf{N}(\mathrm{Sch}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

which satisfies descent with respect to proper hypercovers and whose restriction to $\mathrm{Sm}_{\mathbb{C}}$ *is equivalent to the functor* $\mathcal{D}_*$ *constructed in the previous subsection.*

There is also a unique symmetric monoidal functor

$$\mathcal{D}^* : \mathbf{N}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

which satisfies descent with respect to proper hypercovers and whose restriction to $\mathrm{Sm}_{\mathbb{C}}$ *is equivalent to the functor* $\mathcal{D}^*$ *constructed in the previous subsection.*

Proof. Let $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$ be the Ind-category of the ∞-category of mixed Hodge complexes. This is a stable presentable ∞-category. We first prove that the composite

$$\mathcal{D}_* : \mathbf{N}(\mathrm{Sm}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$$

satisfies descent with respect to proper hypercovers. Let Y be a smooth scheme and $X_\bullet \to Y$ be a hypercover for the proper topology. We wish to prove that the map

$$\alpha : \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathcal{D}_*(X_\bullet) \longrightarrow \mathcal{D}_*(Y)$$

is an equivalence in $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$. By [Bla16, Proposition 3.24] and the fact that taking singular chains commutes with homotopy colimits in spaces, we see that the map

$$\beta : \mathrm{colim}_{\Delta^{\mathrm{op}}} S_*(X_\bullet, \mathbb{Q}) \longrightarrow S_*(Y, \mathbb{Q})$$

is an equivalence. On the other hand, writing $\mathbf{Ch}_*(\mathbb{Q})^\omega$ for the ∞-category of chain complexes whose homology is finite dimensional, the forgetful functor

$$U : \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}}) \longrightarrow \mathrm{Ind}(\mathbf{Ch}_*(\mathbb{Q})^\omega) \simeq \mathbf{Ch}_*(\mathbb{Q})$$

preserves colimits and by Theorem 6.4, the composite $U \circ \mathcal{D}_*$ is weakly equivalent to $S_*(-, \mathbb{Q})$. Therefore, the map β is weakly equivalent to the map $U(\alpha)$ in particular, we deduce that the source of α is in $\mathbf{MHC}_{\mathbb{Q}}$ (as opposed to $\mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$). And since the functor $U : \mathbf{MHC}_{\mathbb{Q}} \to \mathbf{Ch}_*(\mathbb{C})$ is conservative, it follows that α is an equivalence as desired.

Hence, by Proposition 6.5, there is a unique extension of $\mathcal{D}_*$ to a symmetric monoidal functor $\mathbf{N}(\mathrm{Sch}_{\mathbb{C}}) \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$ that has proper descent. Moreover, as we proved above, if Y is an object of $\mathrm{Sch}_{\mathbb{C}}$ and $X_\bullet \longrightarrow Y$ is a proper hypercover by smooth schemes, then

$\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathcal{D}_*(X_\bullet, \mathbb{Q})$ has finitely generated homology. It follows that this unique extension of $\mathcal{D}_*$ to $\mathrm{Sch}_{\mathbb{C}}$ lands in $\mathbf{MHC}_{\mathbb{Q}} \subset \mathrm{Ind}(\mathbf{MHC}_{\mathbb{Q}})$.

For the case of $\mathcal{D}^*$, since dualization induces a symmetric monoidal equivalence of ∞-categories $\mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \simeq \mathbf{MHC}_{\mathbb{Q}}$, we see that we have no other choice but to define $\mathcal{D}^*$ as the composite

$$\mathbf{N}(\mathrm{Sch})^{\mathrm{op}} \xrightarrow{(\mathcal{D}_*)^{\mathrm{op}}} \mathbf{MHC}_{\mathbb{Q}}^{\mathrm{op}} \xrightarrow{(-)^\vee} \mathbf{MHC}_{\mathbb{Q}}$$

and this will be the unique symmetric monoidal functor

$$\mathcal{D}^* : \mathbf{N}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{MHC}_{\mathbb{Q}}$$

which satisfies descent with respect to proper hypercovers and whose restriction to $\mathrm{Sm}_{\mathbb{C}}$ is equivalent to the functor $\mathcal{D}^*$ constructed in the previous subsection. □

Proposition 6.7. *(1) There is a weak equivalence $\mathcal{D}_*(-)_{\mathbb{Q}} \simeq S_*(-, \mathbb{Q})$ in the category of symmetric monoidal ∞-functors $\mathbf{N}(\mathrm{Sch}_{\mathbb{C}}) \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$.*
(2) There is a weak equivalence $\mathcal{A}^*_{PL}(-) \simeq \mathcal{D}^*(-)_{\mathbb{Q}} \simeq S^*(-, \mathbb{Q})$ in the category of symmetric monoidal ∞-functors $\mathbf{N}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$.

Proof. We prove the first claim. By construction $\mathcal{D}_*(-)_{\mathbb{Q}}$ is a symmetric monoidal functor that satisfies proper descent. By [Bla16, Proposition 3.24], the same is true for $S_*(-, \mathbb{Q})$. Since these two functors are moreover weakly equivalent when restricted to $\mathrm{Sm}_{\mathbb{C}}$, they are equivalent by Proposition 6.5.

The linear dual functor is strong monoidal when restricted to chain complexes whose homology is of finite type. Moreover, both $S_*(-, \mathbb{Q})$ and $\mathcal{D}_*(-)_{\mathbb{Q}}$ land in the ∞-category of such chain complexes. Therefore, the equivalence $S^*(-, \mathbb{Q}) \simeq \mathcal{D}^*(-)_{\mathbb{Q}}$ follows from the first part. The equivalence $\mathcal{A}^*_{PL}(-) \simeq S^*(-, \mathbb{Q})$ is classical. □

7. Formality of the singular chains functor

In this section, we prove the main results of the paper on the formality of the singular chains functor. We also explain some applications to operad formality.

Definition 7.1. Let X be a complex scheme and let α be a rational number. We say that the weight filtration on $H^*(X, \mathbb{Q})$ is *α-pure* if for all $n \geq 0$ we have

$$Gr_p^W H^n(X, \mathbb{Q}) = 0 \text{ for all } p \neq \alpha n.$$

Remark 7.2. Note that since the weight filtration on $H^n(-, \mathbb{Q})$ has weights in the interval $[0, 2n] \cap \mathbb{Z}$, the above definition makes sense only for $\alpha \in [0, 2] \cap \mathbb{Q}$. For $\alpha = 1$ we recover the purity property shared by the cohomology of smooth projective varieties. A very simple example of a variety whose filtration is α-pure, with α not integer, is given by $\mathbb{C}^2 \setminus \{0\}$. Its cohomology is concentrated in degree 3 and weight 4, so its weight filtration is 4/3-pure. We refer to Proposition 8.2 in the following section for more elaborate examples.

Here is our main theorem.

Theorem 7.3. *Let α be a non-zero rational number. The singular chains functor*

$$S_*(-, \mathbb{Q}) : \mathrm{Sch}_{\mathbb{C}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to schemes whose weight filtration in cohomology is α-pure.

Proof. By Corollay 2.4, it suffices to prove that this functor is formal as an ∞-lax monoidal functor. By Proposition 6.7, it is equivalent to prove that $\mathcal{D}_*(-)_\mathbb{Q}$ is formal. We denote by $\bar{\mathcal{D}}_*$ the composite of $\mathcal{D}_*$ with a symmetric monoidal inverse of the equivalence of Theorem 5.4. Because of that theorem, $\mathcal{D}_*(-)_\mathbb{Q}$ is weakly equivalent to $\Pi_\mathbb{Q} \circ \bar{\mathcal{D}}_*$. The restriction of $\bar{\mathcal{D}}_*$ to $\mathrm{Sch}_\mathbb{C}^{\alpha\text{-}pure}$ lands in $\mathbf{Ch}_*(\mathrm{MHS}_\mathbb{Q})^{\alpha\text{-}pure}$, the full subcategory of $\mathbf{Ch}_*(\mathrm{MHS}_\mathbb{Q})$ spanned by chain complexes whose homology is α-pure. By Corollary 4.6, the ∞-functor $\Pi_\mathbb{Q}$ from $\mathbf{Ch}_*(\mathrm{MHS}_\mathbb{Q})^{\alpha\text{-}pure}$ to $\mathbf{Ch}_*(\mathbb{Q})$ is formal and hence so is $\Pi_\mathbb{Q} \circ \bar{\mathcal{D}}_*$. □

We now list a few applications of this result.

7.1. **Noncommutative little disks operad.** The authors of [DSV15] introduce two non-symmetric topological operads $\mathcal{A}s_{S^1}$ and $\mathcal{A}s_{S^1} \rtimes S^1$. In each arity, these operads are given by a product of copies of $\mathbb{C} - \{0\}$ and the operad maps can be checked to be algebraic maps. It follows that the operads $\mathcal{A}s_{S^1}$ and $\mathcal{A}s_{S^1} \rtimes S^1$ are operads in the category $\mathrm{Sm}_\mathbb{C}$ and that the weight filtration on their cohomology is 2-pure. Therefore, by 7.3 we have the following result.

Theorem 7.4. *The operads $S_*(\mathcal{A}s_{S^1}, \mathbb{Q})$ and $S_*(\mathcal{A}s_{S^1} \rtimes S^1, \mathbb{Q})$ are formal.*

Remark 7.5. The fact that the operad $S_*(\mathcal{A}s_{S^1}, \mathbb{Q})$ is formal is proved in [DSV15, Proposition 7] by a more elementary method and it is true even with integral coefficients. The other formality result was however unknown to the authors of [DSV15].

7.2. **Self-maps of the projective line.** We denote by F_d the algebraic variety of degree d algebraic maps from $\mathbf{P}^1_\mathbb{C}$ to itself that send the point ∞ to the point 1. Explicitly, a point in F_d is a pair (f, g) of degree d monic polynomials without any common roots. Sending a monic polynomials to its set coefficients, we may see the variety F_d as a Zariski open subset of $\mathbf{A}^{2d}_\mathbb{C}$. See [Hor16, Section 5] for more details.

Proposition 7.6. *The weight filtration on $H^*(F_d, \mathbb{Q})$ is 2-pure.*

Proof. The variety F_d is denoted $\mathrm{Poly}_1^{d,2}$ in [FW16, Definition 1.1.]. It is explained in Step 4 of the proof of Theorem 1.2. in that paper that the variety F_d is the quotient of the complement of a hyperplane arrangement H in $\mathbf{A}^{2d}_\mathbb{C}$ by the group $\Sigma_d \times \Sigma_d$ acting by permuting the coordinates. A transfer argument then shows that $H^k(F_d, \mathbb{Q})$ is a subspace of $H^k(\mathbf{A}^{2d}_\mathbb{C} - H, \mathbb{Q})$. Moreover this inclusion is a morphism of mixed Hodge structures. Since the mixed Hodge structure of $H^k(\mathbf{A}^{2d}_\mathbb{C} - H, \mathbb{Q})$ is well-known to be pure of weight $2k$ (by Proposition 8.2 or by [Kim94]), the desired result follows. □

In [Caz12, Proposition 3.1.], Cazanave shows that the scheme $\bigsqcup_d F_d$ has the structure of a graded monoid in $\mathrm{Sm}_\mathbb{C}$. The structure of a graded monoid can be encoded by a colored operad. Thus the following follows from 7.3.

Theorem 7.7. *The graded monoid in chain complexes $\bigoplus_d S_*(F_d, \mathbb{Q})$ is formal.*

7.3. **The little disks operad.** In [Pet14], Petersen shows that the operad of little disks $\mathcal{D}$ is formal. The method of proof is to use the action of a certain group $\mathrm{GT}(\mathbb{Q})$ on $S_*(\mathcal{PAB}_\mathbb{Q}, \mathbb{Q})$ which follows from work of Drinfeld's. Here the operad $\mathcal{PAB}_\mathbb{Q}$ is rationally equivalent to $\mathcal{D}$ and $\mathrm{GT}(\mathbb{Q})$ is the group of $\mathbb{Q}$-points of the pro-algebraic Grothendieck-Teichmüller group. We can reinterpret this proof using the language of mixed Hodge structures. Indeed, the group GT receives a map from the group $\mathrm{Gal}(\mathrm{MT}(\mathbb{Z}))$, the Galois group of the Tannakian category of mixed Tate motives over $\mathbb{Z}$ (see [And04, 25.9.2.2]). Moreover there is a map $\mathrm{Gal}(\mathrm{MHTS}_\mathbb{Q}) \to \mathrm{Gal}(\mathrm{MT}(\mathbb{Z}))$ from the Tannakian Galois group of the abelian category of mixed Hodge Tate structures (the full subcategory of $\mathrm{MHS}_{\mathbb{Q}}$ generated under extensions by the Tate twists $\mathbb{Q}(n)$ for all n) which is Tannaka dual to the tensor functor

$$\mathrm{MT}(\mathbb{Z}) \longrightarrow \mathrm{MHTS}_{\mathbb{Q}}$$

sending a mixed Tate motive to its Hodge realization. This map of Galois group allows us to view $S_*(\mathcal{PAB}_{\mathbb{Q}}, \mathbb{Q})$ as an operad in $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$ which moreover has a 2-pure weight filtration (as follows from the computation in [Pet14]). Therefore by Corollary 4.6, the operad $S_*(\mathcal{PAB}_{\mathbb{Q}}, \mathbb{Q})$ is formal and hence also $S_*(\mathcal{D}, \mathbb{Q})$.

7.4. **The gravity operad.** In [DH17], Dupont and the second author prove the formality of the gravity operad of Getzler. It is an operad structure on the collection of graded vector spaces $\{H_{*-1}(\mathcal{M}_{0,n+1}), n \in \mathbb{N}\}$. It can be defined as the homotopy fixed points of the circle action on $S_*(\mathcal{D}, \mathbb{Q})$. The method of proof in [DH17] can also be interpreted in terms of mixed Hodge structures. Indeed, a model $\mathcal{G}rav^{W'}$ of gravity is constructed in 2.7 of loc. cit. This model comes with an action of $\mathrm{GT}(\mathbb{Q})$ and a $\mathrm{GT}(\mathbb{Q})$-equivariant map $\iota : \mathcal{G}rav^{W'} \longrightarrow S_*(\mathcal{PAB}_{\mathbb{Q}}, \mathbb{Q})$ which is injective on homology. As in the previous subsection, this action of $\mathrm{GT}(\mathbb{Q})$ lets us interpret $\mathcal{G}rav^{W'}$ as an operad in $\mathrm{Ch}_*(\mathrm{MHS}_{\mathbb{Q}})$. Moreover, the injectivity of ι implies that $\mathcal{G}rav^{W'}$ also has a 2-pure weight filtration. Therefore by Corollary 4.6, we deduce the formality of $\mathcal{G}rav^{W'}$. In fact, we obtain the stronger result that the map

$$\iota : \mathcal{G}rav^{W'} \longrightarrow S_*(\mathcal{PAB}_{\mathbb{Q}}, \mathbb{Q})$$

is formal as a map of operads (i.e. it is connected to the induced map in homology by a zig-zag of maps of operads).

7.5. E^1**-formality.** The above results deal with objects whose weight filtration is pure. In general, for mixed weights, the singular chains functor is not formal, but it is E^1-formal as we now explain.

The r-stage of the spectral sequence associated to a filtered complex is an r-bigraded complex with differential of bidegree $(-r, r-1)$. By taking its total degree and considering the column filtration we obtain a filtered complex. Denote by

$$E^r : \mathbf{Ch}_*(\mathcal{F}\mathbb{Q}) \longrightarrow \mathbf{Ch}_*(\mathcal{F}\mathbb{Q})$$

the resulting symmetric monoidal ∞-functor. Denote by

$$\tilde{\Pi}^W_{\mathbb{Q}} : \mathbf{MHC}_{\mathbb{Q}} \longrightarrow \mathbf{Ch}_*(\mathcal{F}\mathbb{Q})$$

the forgetful functor defined by sending a mixed Hodge complex to its rational component together with the weight filtration. Note that, since the weight spectral sequence of a mixed Hodge complex degenerates at the second stage, the homology of $E^1 \circ \tilde{\Pi}^W_{\mathbb{Q}}$ gives the weight filtration on the homology of mixed Hodge complexes. We have:

Theorem 7.8. *Denote by* $S_*^{fil} : \mathbf{N}(\mathrm{Sch}_{\mathbb{C}}) \longrightarrow \mathbf{Ch}_*(\mathbb{Q})$ *the composite functor*

$$\mathbf{N}(\mathrm{Sch}_{\mathbb{C}}) \xrightarrow{\mathcal{D}_*} \mathbf{MHC}_{\mathbb{Q}} \xrightarrow{\tilde{\Pi}^W_{\mathbb{Q}}} \mathbf{Ch}_*(\mathcal{F}\mathbb{Q}).$$

There is an equivalence of symmetric monoidal ∞*-functors* $E^1 \circ S_*^{fil} \simeq S_*^{fil}$.

Proof. It suffices to prove an equivalence $\tilde{\Pi}^W_{\mathbb{Q}} \simeq E^1 \circ \tilde{\Pi}^W_{\mathbb{Q}}$. We have a commutative diagram of symmetric monoidal ∞-functors.

$$\begin{array}{ccc}
\mathbf{Ch}_*(\mathrm{MHS}_{\mathbb{Q}}) & \xrightarrow{\mathcal{T}} & \mathbf{MHC}_{\mathbb{Q}} \\
\Big\downarrow{\scriptstyle \Pi^W_{\mathbb{Q}}} & & \Big\downarrow{\scriptstyle \tilde{\Pi}^W_{\mathbb{Q}}} \\
\mathbf{Ch}_*(\mathcal{F}\mathbb{Q}) & \xrightarrow{T} & \mathbf{Ch}_*(\mathcal{F}\mathbb{Q}) \\
\Big\downarrow{\scriptstyle E^0} & & \Big\downarrow{\scriptstyle E^1} \\
\mathbf{Ch}_*(\mathcal{F}\mathbb{Q}) & \xrightarrow{T} & \mathbf{Ch}_*(gr\mathbb{Q})
\end{array}$$

The commutativity of the top square follows from the definition of $\mathcal{T}$. We prove that the bottom square commutes. Recall that $T(K,W)$ is the filtered complex (K,TW) defined by $TW^pK_n := W^{p+n}K_n$. It satisfies $d(TW^pK_p) \subset TW^{p+1}K_{n-1}$. In particular, the induced differential on $Gr_{TW}K$ is trivial. Therefore we have:

$$E^1_{-p,q}(K,TW) \cong H_{q-p}(Gr^p_{TW}K) \cong Gr^p_{TW}K_{q-p} = Gr^q_W K_{q-p} = E^0_{-q,2q-p}(K,W).$$

This proves that the above diagram commutes.

Since $\mathcal{T}$ is an equivalence of ∞-categories, it is enough to prove that $E^1 \circ \tilde{\Pi}^W_{\mathbb{Q}} \circ \mathcal{T}$ is equivalent to $\tilde{\Pi}^W_{\mathbb{Q}} \circ \mathcal{T}$. By the commutation of the above diagram it suffices to prove that there is an equivalence $E^0 \circ \Pi^W_{\mathbb{Q}} \cong \Pi^W_{\mathbb{Q}}$. This follows from Lemma 4.4, since $E^0 = U^{fil} \circ gr$. □

8. Rational homotopy of schemes and formality

For X a space, we denote by $\mathcal{A}^*_{PL}(X)$, Sullivan's algebra of piecewise linear differential forms. This is a commutative dg-algebra over $\mathbb{Q}$ that captures the rational homotopy type of X. A contravariant version of Theorem 7.3 gives:

Theorem 8.1. *Let α be a non-zero rational number. The functor*

$$\mathcal{A}^*_{PL} : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{C}} \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$$

is formal as a lax symmetric monoidal functor when restricted to schemes whose weight filtration in cohomology is α-pure.

Proof. The proof is the same as the proof of Theorem 7.3 using $\mathcal{D}^*$ instead of $\mathcal{D}_*$ and using the fact that $\mathcal{D}^*(-)_{\mathbb{Q}}$ is quasi-isomorphic to $\mathcal{A}^*_{PL}$ as a lax monoidal functor (see [NA87, Théorème 5.5]). □

Recall that a topological space X is said to be *formal* if there is a string of quasi-isomorphisms of commutative dg-algebras from $\mathcal{A}^*_{PL}(X)$ to $H^*(X,\mathbb{Q})$, where $H^*(X,\mathbb{Q})$ is considered as a commutative dg-algebra with trivial differential. Likewise, a continuous map of topological spaces $f : X \longrightarrow Y$ is *formal* if there is a string of homotopy commutative diagrams of morphisms

$$\begin{array}{ccccccccc}
\mathcal{A}^*_{PL}(Y) & \longleftarrow & * & \longleftarrow & \cdots & \longrightarrow & * & \longrightarrow & H^*(Y,\mathbb{Q}) \\
\Big\downarrow{\scriptstyle f^*} & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow{\scriptstyle H^*(f)} \\
\mathcal{A}^*_{PL}(X) & \longleftarrow & * & \longleftarrow & \cdots & \longrightarrow & * & \longrightarrow & H^*(X,\mathbb{Q})
\end{array}$$

where the horizontal arrows are quasi-isomorphisms. Note that if $f : X \to Y$ is a map of topological spaces and X and Y are both formal spaces, then it is not always true that f is a formal map. Also, in general, the composition of formal morphisms is not formal.

Theorem 8.1 gives functorial formality for schemes with pure weight filtration in cohomology, generalizing both "purity implies formality" statements appearing in [Dup16] for smooth varieties and in [CC17] for singular projective varieties. We also get a result of partial formality as done in these references, via Proposition 2.10. Our generalization is threefold, as explained in the following three subsections.

8.1. **Rational weights.** To our knowledge, in the existing references where α-purity of the weight filtration is discussed, only the cases $\alpha = 1$ and $\alpha = 2$ are considered, whereas we obtain formality for varieties with α-pure cohomology, for α an arbitrary non-zero rational number. This gives a whole new family of formal spaces. For instance, we have:

Proposition 8.2. *Let $H = \{H_1, \dots, H_k\}$ be a set of linear subspaces of $\mathbb{C}^n$ such that for all proper subset $S \subset \{1, \dots, k\}$, the intersection $H_S := \cap_{i\in S} H_i$ is of codimension $d|S|$. Then the mixed Hodge structure on $H^*(\mathbb{C}^n - \cup_i H_i, \mathbb{Q})$ is pure of weight $2d/(2d-1)$.*

Proof. We proceed by induction on k. This is easy to do for $k = 1$. Now, we consider the variety $X = \mathbb{C}^n - \cup_i^{k-1} H_i$, It contains an open subvariety $U = \mathbb{C}^n - \cup_i^k H_i$ and its closed complement $Z = H_k - \cup_i^{k-1} H_i \cap H_k$ which has codimension d. Therefore the purity long exact sequence on cohomology groups has the form

$$\dots \longrightarrow H^{r-2d}(Z)(-d) \longrightarrow H^r(X) \longrightarrow H^r(U) \longrightarrow H^{r+1-2d}(Z)(-d) \longrightarrow \dots$$

By the induction hypothesis, the Hodge structure on $H^{r+1-2d}(Z)(-d)$ and on $H^r(X)$ are pure of weight $2dr/(2d-1)$ and hence it is also the case for $H^r(U)$ as desired. □

Remark 8.3. This proposition is well-known for $d = 1$ and is proved for instance in [Kim94].

8.2. **Functoriality.** Every morphism of smooth complex projective schemes is formal. However, if $f : X \to Y$ is an algebraic morphism of complex schemes (possibly singular and/or non-projective), and both X and Y are formal, the morphism f need not be formal.

Example 8.4. Consider the algebraic Hopf fibration $f : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbf{P}^1_{\mathbb{C}}$ defined by $(x_0, x_1) \mapsto [x_0 : x_1]$. Both spaces $\mathbb{C}^2 \setminus \{0\} \simeq S^3$ and $\mathbf{P}^1_{\mathbb{C}} \simeq S^2$ are formal. As is well-known, the morphism induced by f in cohomology is trivial, while its homotopy type is not. Therefore f is not formal. Note in fact, that $\mathbf{P}^1_{\mathbb{C}}$ has 1-pure weight filtration while $\mathbb{C}^2 \setminus \{0\}$ has 4/3-pure weight filtration.

Theorem 8.1 tells us that if $f : X \longrightarrow Y$ is a morphism of algebraic varieties and both X and Y have α-pure cohomology, with α a non-zero rational number (the same α for X and Y), then f is a formal morphism. This generalizes the formality of holomorphic morphisms between compact Kähler manifolds of [DGMS75] and enhances the results of [Dup16] and [CC17] by providing them with functoriality. In fact, we have:

Proposition 8.5. *Let $f : X \longrightarrow Y$ be a morphism of complex schemes inducing a non-trivial morphism $f^* : \mathcal{A}^*_{PL}(Y) \longrightarrow \mathcal{A}^*_{PL}(X)$. Assume that the weight filtration on the cohomology of X (resp. Y) is α-pure (resp. β-pure). Then:*

(1) If $\alpha \neq \beta$ then $H^*(f) = 0$.
(2) f is formal if and only if $\alpha = \beta$.

Proof. Assume that $\alpha \neq \beta$. Since morphisms of mixed Hodge structures are strictly compatible with the weight filtration, to show that $H^*(f)$ is trivial it suffices to show that the morphism

$$Gr_p^W H^n(Y, \mathbb{Q}) \longrightarrow Gr_p^W H^n(X, \mathbb{Q})$$

is trivial for all $p \in \mathbb{Z}$ and all $n > 0$. This follows from the purity conditions. Now, since f^* is assumed to be non-trivial and $H^*(f) = 0$, it follows that f cannot be formal. To end the proof, note that when $\alpha = \beta$, Theorem 8.1 ensures that f is formal. □

8.3. **Non-projective singular schemes.** The formality of non-projective singular complex varieties with pure Hodge structure seems to be a new result.

Example 8.6. Let X be an irreducible singular projective variety of dimension n with 1-pure weight filtration in cohomology (for instance, a V-manifold). Let $p \in X$ be a smooth point of X. Then the complement $X - p$ has 1-pure weight filtration in cohomology. Indeed, using a Mayer-Vietoris long exact sequence argument, one can show that for $k \leq 2n-1$ we have $H^k(X-p) \cong H^k(X)$ and $H^{2n}(X-p) = 0$. Therefore the inclusion $X - p \hookrightarrow X$ is formal.

8.4. E_1**-formality.** We also have a contravariant version of Theorem 7.8.

Theorem 8.7. *Denote by* $\mathcal{A}^*_{fil} : \mathbf{N}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \longrightarrow \mathbf{Ch}_*(\mathcal{F}\mathbb{Q})$ *the composite functor*

$$\mathbf{N}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \xrightarrow{\mathcal{D}^*} \mathbf{MHC}_{\mathbb{Q}} \xrightarrow{\tilde{\Pi}^W_{\mathbb{Q}}} \mathbf{Ch}_*(\mathcal{F}\mathbb{Q}).$$

Then

(1) *The lax symmetric monoidal ∞-functors $\mathcal{A}^*_{fil}$ and $E_1 \circ \mathcal{A}^*_{fil}$ are weakly equivalent.*
(2) *Let $U : \mathrm{Ch}_*(\mathcal{F}\mathbb{Q}) \longrightarrow \mathrm{Ch}_*(\mathbb{Q})$ denote the forgetful functor. The lax symmetric monoidal ∞-functor $U \circ E_1 \circ \mathcal{A}^*_{fil} : \mathbf{N}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{op}} \to \mathbf{Ch}_*(\mathbb{Q})$ is weakly equivalent to Sullivan's functor $\mathcal{A}^*_{PL}$ of piece-wise linear forms.*
(3) *The lax symmetric monoidal functor $U \circ E_1 \circ \mathcal{A}^*_{fil} : \mathrm{Sm}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{Ch}_*(\mathbb{Q})$ is weakly equivalent to Sullivan's functor $\mathcal{A}^*_{PL}$ of piece-wise linear forms.*

Proof. The first part is proven as Theorem 7.8 replacing $\mathcal{D}^*$ by $\mathcal{D}_*$. The second part follows from the first part and the fact that $\mathcal{A}^*_{PL}(-)$ is naturally weakly equivalent to $\mathcal{D}^*(-)_{\mathbb{Q}} \simeq U \circ \mathcal{A}^*_{fil}$ (Proposition 6.7). The third part follows from the second part and Theorem 2.3, using the fact that both functors are ordinary lax monoidal functors when restricted to smooth schemes. □

Remark 8.8. In [Mor78] it is proven that the complex homotopy type of every smooth complex scheme is E_1-formal. This is extended to possibly singular schemes and their morphisms in [CG14]. Then, a descent argument is used to prove that for nilpotent spaces (with finite type minimal models), this result descends to the rational homotopy type. Theorem 8.7 enhances the contents of [CG14] in two ways: first, since descent is done at the level of functors, we obtain E_1-formality over $\mathbb{Q}$ for any complex scheme, without nilpotency conditions (the only property needed is finite type cohomology). Second, the functorial nature of our statement makes E_1-formality at the rational level, compatible with composition of morphisms.

8.5. **Formality of Hopf cooperads.** Our main theorem takes two dual forms, one covariant and one contravariant. The covariant theorem yields formality for algebraic structures (like monoids, operads, etc.), the contravariant theorem yields formality for coalgebraic structure (like the comonoid structure coming from the diagonal $X \to X \times X$ for any variety X). One might wonder if there is a way to do both at the same time. For example, if M is a topological monoid, then $H^*(M, \mathbb{Q})$ is a Hopf algebra where the multiplication comes from the diagonal of M and the comultiplication comes from the multiplication of M. One may ask wether $S^*(M, \mathbb{Q})$ is formal as a Hopf algebra. This question is not well-posed because $S^*(M, \mathbb{Q})$ is not a Hopf algebra on the nose. The problem is that there does not seem to exist a model for singular chains or cochains that is strong monoidal : the standard singular chain functor $S_*(-, \mathbb{Q})$ is lax monoidal and Sullivan's functor $\mathcal{A}^*_{PL}$ is oplax monoidal functor from Top to $\mathrm{Ch}_*(\mathbb{Q})^{\mathrm{op}}$.

Nevertheless, the functor $\mathcal{A}^*_{PL}$ is strong monoidal "up to homotopy". It follows that, if M is a topological monoid, $\mathcal{A}^*_{PL}(M)$ has the structure of a cdga with a comultiplication up to homotopy and it makes sense to ask if it has formality as such an object. In order to formulate this more precisely, we introduce the notion of an algebraic theory. The following is inspired by Section 3 of [LV14].

Definition 8.9. An algebraic theory is a small category T with finite products. For $\mathcal{C}$ a category with finite products, a T-algebra in $\mathcal{C}$ is a finite product preserving functor $T \longrightarrow \mathcal{C}$.

There exist algebraic theories for which the T-algebras are monoids, groups, rings, operads, cyclic operads, modular operads etc.

Remark 8.10. Definitions of algebraic theories in the literature are usually more restrictive. This definition will be sufficient for our purposes.

Definition 8.11. Let T be an algebraic theory. Let $\Bbbk$ be a field. Then a dg Hopf T-coalgebra over $\Bbbk$ is a finite coproduct preserving functor from T^{op} to the category of cdga's over $\Bbbk$.

Remark 8.12. Recall that the coproduct in the category of cdga's is the tensor product. It follows that a dg Hopf T-algebra for T the algebraic theory of monoids is a dg Hopf algebra whose multiplication is commutative. A dg Hopf T-algebra for T the theory of operads is what is usually called a dg Hopf cooperad in the literature.

Definition 8.13. Let T be an algebraic theory and $\mathcal{C}$ a category with products and with a notion of weak equivalences. A weak T-algebra in $\mathcal{C}$ is a functor $F : T \longrightarrow \mathcal{C}$ such that for each pair (s, t) of objects of T, the canonical map

$$F(t \times s) \longrightarrow F(t) \times F(s)$$

is a weak equivalence. A weak T-algebra in the opposite category of $\mathrm{CDGA}_\Bbbk$ is called a weak dg Hopf T-coalgebra.

Observe that if $X : T \longrightarrow \mathrm{Top}$ is a T-algebra in topogical spaces (or even a weak T-algebra), then $\mathcal{A}^*_{PL}(X)$ is a weak dg Hopf T-coalgebra. Our main theorem for Hopf T-coalgebras is the following.

Theorem 8.14. *Let α be a rational number different from zero. Let $X : T \longrightarrow \mathrm{Sch}_\mathbb{C}$ be a T-algebra such that for all $t \in T$, the weight filtration on the cohomology of $X(t)$ is α-pure. Then $\mathcal{A}^*_{PL}(X)$ is formal as a weak dg Hopf T-coalgebra.*

Proof. Being a weak T-coalgebra is a property of a functor $T^{\mathrm{op}} \to \mathrm{CDGA}_\Bbbk$ that is invariant under quasi-isomorphism. Thus the result follows immediately from Theorem 8.1. □

It should be noted that knowing that $\mathcal{A}^*_{PL}(X)$ is formal as a dg Hopf T-coalgebra implies that the data of $H^*(X, \mathbb{Q})$ is enough to reconstruct X as a T-algebra up to rational equivalence. Indeed, recall the Sullivan spatial realization functor

$$\langle - \rangle : \mathrm{CDGA}_\Bbbk \longrightarrow \mathrm{Top}$$

Applying this functor to a weak dg Hopf T-coalgebra yields a weak T-algebra in rational spaces. Specializing to $\mathcal{A}^*_{PL}(X)$ where X is a T-algebra in spaces, we get a rational model for X in the sense that the map

$$X \longrightarrow \langle \mathcal{A}^*_{PL}(X) \rangle$$

is a rational weak equivalence of weak T-algebras whose target is objectwise rational. It should also be noted that for reasonable algebraic theories T (including in particular the theory for monoids, commutative monoids, operads, cyclic operads), the homotopy theory of T-algebras in spaces is equivalent to that of weak T-algebras by the main theorem of [Ber06]. In particular our weak T-algebra $\langle \mathcal{A}^*_{PL}(X) \rangle$ can be stricitified to a strict T-algebra that models the rationalization of X. If $\mathcal{A}^*_{PL}(X)$ is formal, one also get a rational model for X by applying the spatial realization to the strict Hopf T-coalgebra $H^*(X, \mathbb{Q})$. Thus the rational homotopy type of X as a T-algebra is a formal consequence of $H^*(X, \mathbb{Q})$ as a Hopf T-coalgebra.

Example 8.15. Applying this theorem to the non-commutative little disks operad and framed little disks operad of subsection 7.1, we deduce that $\mathcal{A}^*_{PL}(\mathcal{A}s_{S^1})$ and $\mathcal{A}^*_{PL}(\mathcal{A}s_{S^1} \rtimes S^1)$ are formal as a weak Hopf non-symmetric cooperads. Similarly applying this to the monoid of self maps of the projective line of subsection 7.2, we deduce that $\mathcal{A}^*_{PL}(\bigsqcup_d F_d)$ is formal as a weak Hopf graded comonoid.

References

[And04] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2004.

[BD78] R. Body and R. Douglas. Rational homotopy and unique factorization. *Pacific J. Math.*, 75(2):331–338, 1978.

[Beĭ86] A. A. Beĭlinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

[Ber06] J. Bergner. Rigidification of algebras over multi-sorted theories. *Algebraic & Geometric Topology*, 6(4):1925–1955, 2006.

[Bla16] A. Blanc. Topological K-theory of complex noncommutative spaces. *Compositio Mathematica*, 152(03):489–555, 2016.

[BMSS98] R. Body, M. Mimura, H. Shiga, and D. Sullivan. p-universal spaces and rational homotopy types. *Comment. Math. Helv.*, 73(3):427–442, 1998.

[BNT15] U. Bunke, T. Nikolaus, and G. Tamme. The Beilinson regulator is a map of ring spectra. *ArXiv:1509.05667*, 2015.

[Bor91] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[BT15] U. Bunke and G. Tamme. Regulators and cycle maps in higher-dimensional differential algebraic K-theory. *Adv. Math.*, 285:1853–1969, 2015.

[Caz12] C. Cazanave. Algebraic homotopy classes of rational functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(4):511–534 (2013), 2012.

[CC17] D. Chataur and J. Cirici. Rational homotopy of complex projective varieties with normal isolated singularities. *Forum Math.*, 29(1):41–57, 2017.

[CG14] J. Cirici and F. Guillén. E_1-formality of complex algebraic varieties. *Algebr. Geom. Topol.*, 14(5):3049–3079, 2014.

[CG16] J. Cirici and F. Guillén. Homotopy theory of mixed Hodge complexes. *Tohoku Math. J.*, 68(3):349–375, 2016.

[Del71] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del74] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[DGMS75] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[DH17] C. Dupont and G. Horel. On two chain models for the gravity operad. *ArXiv:1702.02479*, 2017.

[Dre15] B. Drew. Rectification of Deligne's mixed Hodge structures. *ArXiv:1511.08288*, 2015.

[DSV15] V. Dotsenko, S. Shadrin, and B. Vallette. Noncommutative $\bar{M}_{0,n+1}$. *ArXiv:1510.03261*, 2015.

[Dug01] D. Dugger. Universal homotopy theories. *Adv. Math.*, 164(1):144–176, 2001.

[Dup16] C. Dupont. Purity, Formality, and Arrangement Complements. *Int. Math. Res. Not. IMRN*, (13):4132–4144, 2016.

[FW16] B. Farb and J. Wolfson. Topology and arithmetic of resultants, I. *New York J. Math.*, 22:801–821, 2016.

[GNPR05] F. Guillén, V. Navarro, P. Pascual, and A. Roig. Moduli spaces and formal operads. *Duke Math. J.*, 129(2):291–335, 2005.

[GS75] P. Griffiths and W. Schmid. Recent developments in Hodge theory: a discussion of techniques and results. In *Discrete subgroups of Lie groups and applicatons to moduli*, pages 31–127. Oxford Univ. Press, Bombay, 1975.

[Hai87] R. M. Hain. The de Rham homotopy theory of complex algebraic varieties. II. *K-Theory*, 1(5):481–497, 1987.

[Hin15] V. Hinich. Rectification of algebras and modules. *Doc. Math.*, 20:879–926, 2015.

[Hin16] V. Hinich. Dwyer-Kan localization revisited. *Homology Homotopy Appl.*, 18(1):27–48, 2016.

[Hor16] G. Horel. Motivic homological stability for configuration spaces of the line. *Bull. Lond. Math. Soc.*, 48(4):601–616, 2016.

[Hub95] A. Huber. *Mixed motives and their realization in derived categories*, volume 1604 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[Kim94] M. Kim. Weights in cohomology groups arising from hyperplane arrangements. *Proc. Amer. Math. Soc.*, 120(3):697–703, 1994.

[LV14] P. Lambrechts and I. Volić. Formality of the little N-disks operad. *Mem. Amer. Math. Soc.*, 230(1079):viii+116, 2014.

[Mor78] J. W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.

[NA87] V. Navarro-Aznar. Sur la théorie de Hodge-Deligne. *Invent. Math.*, 90(1):11–76, 1987.

[Pet14] D. Petersen. Minimal models, GT-action and formality of the little disk operad. *Selecta Math. (N.S.)*, 20(3):817–822, 2014.

[PS08] C. Peters and J. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2008.

[RR16] B. Rodríguez and A. Roig. Godement resolutions and operad sheaf homotopy theory. *Collect. Math.*, 2016.

[Sul77] D. Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.

[Wat79] W. C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.

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