

Poincaré duality in 6 functors formalism

after Scholze, Zavyalov

intro: Abstracting Poincaré duality with the formalism in 6 functors formalism

in top: thm if X is a compact oriented manifold of dim d ,
 $R\Gamma(X, \mathbb{Z})[d] \simeq R\Gamma(X, \mathbb{Z}) \xleftarrow{\quad} H^{d-*}(X, \mathbb{Z}) \simeq H_*(X, \mathbb{Z})$
 i.e.

1st generalization: for all $A \in D(X)$, $R\Gamma(X, A^\vee)[d] \simeq R\Gamma(X, A)^\vee$

Rk: Serre duality: for qc coherent sheaves
 if X smooth and proper, $H^i(X, A) \simeq H^{d-i}(X, K_X \otimes A^\vee)$ where $\omega_X \simeq \Omega_X^{1,0}$
 $A \in D_{qc}(X)$
 $H^*(X, K_X \otimes A^\vee)[d] \simeq H^*(X, A)^\vee$ where $\omega_X = -\Omega_X^{1,0}[d]$.
 $= H^*(X, \omega_X \otimes A^\vee) \simeq H^*(X, A)^\vee$

2nd generalization: here $(-)^\vee = \text{Hom}(-, \mathbb{Z}) \rightarrow \text{replace } \text{Hom}(-, \mathbb{Z}) \text{ by } \text{Hom}(-, \mathcal{B})$ where $\mathcal{B} \in D(\mathbb{Z})$

$f: X \rightarrow Y$ $R\Gamma(X, \text{Hom}_{\mathcal{O}_X}(A, f^* \mathcal{B})) [d] \simeq \text{Hom}_{D(\mathbb{Z})}(R\Gamma(X, A), \mathcal{B})$

i.e. $\text{Hom}_{D(\mathbb{Z})}(A, f^* \mathcal{B}) [d] \simeq \text{Hom}_{D(\mathbb{Z})}(f_* A, \mathcal{B})$

i.e. $(f_*, f^* (-)[d])$ are adjoint
 $= f^* (-) \otimes \mathbb{Z}[d]$ dualizing sheaf

Rk for Serre duality: $(f_*, f^* \otimes \mathbb{Z}^d [d])$

3rd generalization: (Verdier duality) = relative version with $f: X \rightarrow Y$ proper (compact) and relatively smooth (manifolds)

then $(f_*, f^* \otimes \omega_{X/Y})$ for some sheaf $\omega_{X/Y} \in \mathcal{D}(X)$ (loc iso to $\mathbb{Z}[d]$)

(pk the assumption "X oriented" gave $\omega_X \cong \mathbb{Z}[d]$ globally.)

formalization by Scholze (for perfectoid spaces to replace the notion of smoothness)

fix a 6functor formalism: $\mathcal{D}: \text{Con}(\mathcal{E}, \mathcal{E}) \rightarrow \text{Cat}_{\infty}$

def a map $f: X \rightarrow Y$ in \mathcal{E} is said to be (D-) cohomologically smooth if

- 1) $(f_!, f^* \otimes \omega_f)$ adjoint (then $\omega_f \cong f^!(\mathbb{1}_Y)$) - ($f^!(\mathbb{1}_Y) \cong f^* \xrightarrow{\cong} f^!$)
- 2) $f^!(\mathbb{1}_Y)$ is \mathcal{O} -invertible

3) 1) and 2) works for base change and duality sheaf commutes w/ base change

seems very difficult to prove

now goal: prove that f smooth $\Rightarrow f$ cohom. smooth.

for this: develop an easy criterion that imply cohom. smoothness.

pk open immersions are coh smooth.

because $f^! \cong f^*$ directly

II. Fourier transform makes Poncaré duality easy!

to simplify: $\gamma = *$ so $f: X \rightarrow *$ (pass to slice category)
 E are all maps $\xrightarrow{\text{kernel}}$ cone of vector

a) Reminder about Fourier-Mukai transform with kernel K

spaces X_1, X_2
 $K \in D(X_1 \times X_2)$

$$\phi_K: D(X_1) \rightarrow D(X_2)$$

$$A \mapsto R^! (p_2^* A \otimes K)$$

$$p_1^* A \rightarrow p_1^* A \otimes K$$

functor of 2 var x and y ($e^{2\pi i xy}$)

A : function $f(x)$

$p_1^* A$: function $F(x, y) = f(x)$

$p_1^* A \otimes K = f(x) e^{2\pi i xy}$

$\int f(x) e^{2\pi i xy} dx$ + composite given by convolution

lots of nice maps $D(X_1) \rightarrow D(X_2)$ can be written as FM transform of \bullet kernel K
Lurie

formally: FM defines a functor $L\mathcal{E}_D \rightarrow \text{Cat} \xrightarrow{2\text{-cat}}$ $\text{ho } D(X)$
definition $\text{obj}: X \in \mathcal{E} \mapsto D(X)$ or $d(X)$

morphisms $\text{Hom}_{L\mathcal{E}_D}^{(X_1 \times X_2)}: d(X_1 \times X_2) \mapsto \phi_K: d(X_1) \rightarrow d(X_2)$
 \cup_K
 the FM transform

composite given by convolution

ex 1: $X_1 = X, X_2 = Y \Rightarrow p_1 = \text{id}, p_2 = f: X \rightarrow Y$
 $K = 1_X \in D(X)$
 $\phi_X(A) = f_! (p_1^* A \otimes 1_X) = f_!(A)$

2) $X_1 = Y, X_2 = X \Rightarrow p_1 = f, p_2 = \text{id}$
 $K = L \in D(X)$
 $\phi_L(A) = \text{id}_! (f^* A \otimes L) = f^*(A) \otimes L$

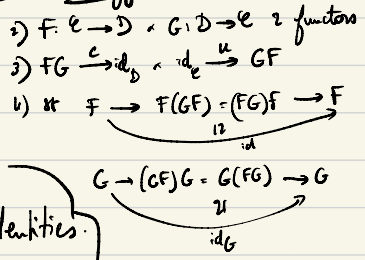
or vice versa we have $(f_!, f^* \circ L)$ are adjoint in Cat . (Cat (a 2cat))
 → enough to check that functors are adjoint in the homom. category.

so it is enough to prove that $1_X \in d(X) = \text{Hom}_{L\mathcal{Z}_D}(X, *) = F$
 and $L \in d(X) = \text{Hom}_{L\mathcal{Z}_D}(*/X) = G$ are adjoint in $L\mathcal{Z}_D$

admit that, like Cat, has the structure of a 2cat (part I)

thm if there is a @ in abble object $L \in d(X)$,
 maps $= \text{id}_X^{L\mathcal{Z}_D} \rightarrow L * 1_X = P_2^* L$
 $\Delta 1_X$
 $f_! L = 1_X * L \rightarrow \text{id}_Y^{L\mathcal{Z}_D} = 1_Y$

reminder Coeffray



or $1_X \rightarrow 1_X * (L * 1_X) = (1_X * L) * 1_X \rightarrow 1_X$
 and $L \rightarrow (L * 1_X) * L = L * (1_X * L) \rightarrow L$ are identities.

then $f_!$ is comologically smooth (review P duality)

Pk the converse is also true (it is an optimal condition)

proof: why $L = f^!(1_Y) \rightarrow$ automatic and base change: true change along $X \xrightarrow{f} Y$ functor $L\mathcal{Z}_D \rightarrow L\mathcal{Z}_D|_{e_{1/Y}}$ preserves adjunction

II. the (co, 2)-variant of $\text{Con}(E, E)$

→ homotopy cat: the 2-cat. LZD

how \mathcal{D} is essentially the Fourier transform: unexpected
 contain

lemma in $\text{Con}(E)$, all objects are self dual

idea if $E = \text{all}$,
 all arrows exist
 in 2 directions
 in correspondence
 → symmetric

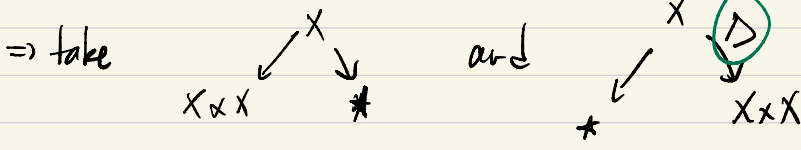
corollary: $\text{Con}(E)$ is self-enriched (closed sym. mon.)

and $\text{Hom}_{\text{Con}}(X, X') = X^\vee \otimes X' = X \otimes X' = X \times X'$

proof of the lemma: we want $X^\vee \otimes X \rightarrow A$ eval map

$A \rightarrow X^\vee \otimes X$ co-eval map.

so we want $X \times X \rightarrow A$
 $A \rightarrow X \times X$



Reminder $\text{LZD} \rightarrow \text{Cat}$
 $X \mapsto d(X)$

$k \in \mathcal{D}(X \times X') = \text{Hom}_{\text{LZD}}(X, X') \mapsto \phi_k: d(X) \rightarrow d(X')$

very similar to $\mathcal{D}: \text{Con}(E, E) \rightarrow \text{Cat}_\infty$

except that $\text{Hom}(X, X') = \mathcal{D}(X \times X') = \mathcal{D}(\text{Hom}_{\text{Con}}(X, X'))$

definition

$\text{Con}(E)^\mathcal{D}$: same as $\text{Con}(E)$
 but the Hom is wrapped along \mathcal{D} lax monoidal
 $(\text{Hom}_{\text{Con}^\mathcal{D}}(X, X') = \mathcal{D}(X \times X'))$

now $\text{Con}(E)^\mathcal{D}$ is enriched in $\text{Cat}_\infty \Rightarrow$ it is an (co, 2) category!

think about

$\mathbb{Z} \rightarrow \text{Cat}$ is the (1,2)-cat deriv of the (0,2)-functor $\tilde{D}: \text{Con}(e)^0 \rightarrow \text{Cat}_\infty$

proof check that Fourier transform

$$\text{Con}(e)^0 \rightarrow \text{Cat}_\infty$$

$$X \mapsto D(X)$$

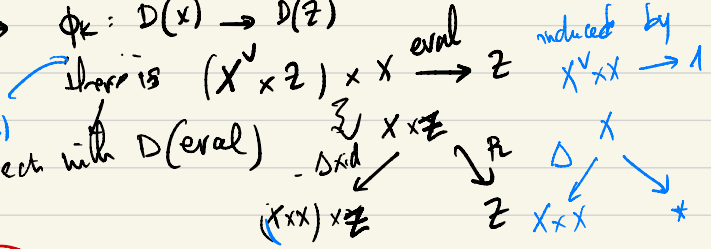
$$X^V = X$$

$$k \in \text{Hom}_{\text{Cat}_\infty}(X, Z) \mapsto \phi_k: D(X) \rightarrow D(Z)$$

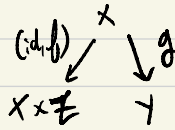
$$= D(X^V \times Z)$$

how does $k \in D(X^V \times Z)$ acts on $D(X)$

check with $D(\text{eval})$



lemma. the correspondence of $f, g: X \rightarrow Z$



$$D \xrightarrow{\quad} D(X) \times D(Z) \rightarrow D(Z)$$

$$(A, B) \mapsto g_! (A \circ f^* B)$$

$$D \xrightarrow{\quad} g_! f^*$$

$$D \left(\begin{array}{ccc} & X \times Z & \\ p_1 \times \text{id} \swarrow & & \downarrow R \\ X \times (X \times Z) & & Z \end{array} \right) \xrightarrow{\quad} D(X \times Z) \times D(X) \rightarrow D(Z)$$

$$(k, \bullet) \mapsto p_2! (k \circ p_1^* B)$$

III. Example: Top.

smooth maps are canon. smooth

→ stably by composite, base change, open immersion
 ⇒ enough to do it for $f: \mathbb{R} \rightarrow *$!

$L = \mathbb{Z}[1]$ now exhibit maps $\alpha: \Delta^1 \mathbb{Z} \rightarrow \mathbb{Z}[1]$ (1)
 $\beta: \text{fib}(L) \rightarrow \mathbb{Z}$ (2)
 $\Gamma_c(\mathbb{R}, \mathbb{Z}(1))$

(2) in fact $\Gamma_c(\mathbb{R}, \mathbb{Z}) \simeq \mathbb{Z}[-1]$ | (1) $\text{fib } \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \Delta^1 \mathbb{Z}$
 \downarrow
 $\text{fib}(\mathbb{R}^2, \mathbb{Z}) \rightarrow \text{fib}(\mathbb{R}, \mathbb{Z}) \rightarrow \text{Hom}(\Delta^1 \mathbb{Z}(1), \mathbb{Z})$
 $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$

= colim (fib $(\mathbb{R}^2(\mathbb{Z}) \rightarrow \mathbb{R}^2(\mathbb{R}[1,1], \mathbb{Z}))$)
 Eriect $\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$

= fib $(\mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z})$
 = $\mathbb{Z}[-1]$

then the comps are isos ^{not id} ⇒ given or adj up to changing α .