Poincaré duality in 6 functors formalism

after Scholze, Zavyalov

intro: Abstracting Poincaré duality until the final form in 6 functor formalism

in Top: thm if X is a compact oriented manifold of dim d,

$$R\Gamma(X,\mathbb{Z})[d] \simeq R\Gamma(X,\mathbb{Z})^\vee \quad \text{ie} \quad H^{d-*}(X,\mathbb{Z}) \simeq H_*(X,\mathbb{Z})$$

1st generalization: for all $A \in D(X)$, $R\Gamma(X, A^\vee)[d] \simeq R\Gamma(X,A)^\vee$

Rk: Serre duality: for qc coherent sheaves

if X smooth and proper, $A \in D_{qc}(X)$

$$H^i(X,A)^\vee \simeq H^{d-i}(X, K_X \otimes A^\vee) \quad \text{where } \omega_X \simeq \Omega^{\wedge n}_X$$

$$H^*(X, K_X \otimes A^\vee)[d] \simeq H^*(X,A)^\vee \qquad \omega_X = \Omega^{nd}_X[d].$$

$$= H^*(X, \omega_X \otimes A^\vee) \simeq H^*(X,A)^\vee$$

here $(-)^\vee = \mathcal{H}om(-,\mathbb{Z}) \to$ replace $\mathcal{H}om(-,\mathbb{Z})$ by $\mathcal{H}om(-,B)$ where $B \in D(\mathbb{Z})$

2nd generalization

$f: X \to *$

$$R\Gamma(X, \mathcal{H}om_{D(X)}(A, f^*B))[d] \simeq \mathcal{H}om_{D(\mathbb{Z})}(R\Gamma(X,A), B)$$

$$\text{ie } \mathcal{H}om_{D(\mathbb{Z})}(A, f^*B[d]) \simeq \mathcal{H}om_{D(\mathbb{Z})}(f_*A, B)$$

ie $(f_*(-), f^*(-)[d])$ are adjoint

$= f^*(-) \otimes \mathbb{Z}[d]$ dualizing sheaf

Rk for Serre duality: $(f_*, f^* \otimes \Omega^d_X[d])$

3rd generalization: (Verdier duality) = relative version with $f: X \to Y$ proper smooth of rel dim d (compact) (manifolds) thm

then $(f_*, f^* \otimes \omega_{X/Y})$ for some sheaf $\omega_{X/Y} \in D(X)$
(loc iso to $\mathbb{Z}[d]$)

(Rk the assumption "X oriented" gave $\omega_X \simeq \mathbb{Z}[d]$ globally.)

formalization by Scholze (for perfectoid spaces to replace the notion of smoothness)

fix a 6 function formalism: $D : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty$

def a morphism $f: X \to Y$ in E is said to be (D-) cohomologically smooth if

1) $(f_!, f^* \otimes \omega_f)$ adjoint (then $\omega_f = f^!(1_Y)$. ($f^!(1_Y) \otimes f^* \xrightarrow{\sim} f^!$))

2) $f^!(1_Y)$ is $\otimes$-invertible

3) 1) and 2) works for base change and dualizing sheaf commutes w/ base change

seems very difficult to prove

now goal: prove that f smooth $\Rightarrow$ f cohom. smooth.

for this: develop an easy criterion that imply cohom. smoothness.

Rk open immersions are coh smooth.
because $f^! \simeq f^*$ directly

II. Fourier transform makes Poincaré duality easy!

to simplify: • $Y = *$ (terminal) so $f : X \to *$ (pass to slice category)

E are all maps → case of Victor

a) Reminder about Fourier–Mukai transform with kernel K

spaces X_1, X_2

$K \in D(X_1 \times X_2)$ ⟷ function of 2 var x and y ($e^{2i\pi xy}$)

$$\phi_K : D(X_1) \longrightarrow D(X_2)$$

$$A \longmapsto p_{2!}(p_1^*A \otimes K)$$

$$A \searrow p_1^*A \to p_1^*A \otimes K \nearrow$$

A : function $f(x)$

p_1^*A : function $F(x,y) = f(x)$

$p_1^*A \otimes K = f(x)\, e^{2i\pi xy}$

$\int f(x)\, e^{2i\pi xy}\, dx$

+ composite given by convolution

lots of nice maps $D(X_1) \to D(X_2)$ can be written as FM transform of a kernel K

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formally definition: FM defines a functor $LZ_D \longrightarrow \mathrm{Cat} \to$ 2 cat, $\nearrow h_0 D(X)$

obj : $X \in \mathcal{C} \longmapsto D(X)$ or $d(X)$

morphisms $\mathrm{Hom}_{LZ_D}(X_1, X_2) = d(X_1 \times X_2) \longmapsto \phi_K : d(X_1) \to d(X_2)$ the FM transform

$\quad K \in d(X_1 \times X_2)$

composite given by convolution

ex: 1) $X_1 = X$, $X_2 = Y \Rightarrow p_1 = id$, $p_2 = f : X \to Y$ $\qquad \phi_{1_X}(A) = f_!(id^*A \otimes 1_X) = f_!(A)$

$K = 1_X \in D(X)$

2) $X_1 = Y$, $X_2 = X$: $p_1 = f$, $p_2 = id$ $\qquad \phi_L(A) = id_!(f^*A \otimes L) = f^*(A) \otimes L$

$K = L \in D(X)$

on veut prouver que $(f_!, f^* \otimes L)$ are adjoint in Cat_∞. Cat (a 2 cat)
→ enough to check that functors are adjoint in the homotopy category.

so it is enough to prove that $1_X \in \mathcal{D}(\underset{}{X}^{X\times *}) = \mathrm{Hom}_{LZ_D}(X, *) = F$
and $L \in \mathcal{D}(X)_{*\times X} = \mathrm{Hom}_{LZ_D}(*, X) = G$ are adjoint in LZ_D

admit that, like Cat, has the structure of a 2 cat (part I)

Reminder Geoffroy.
2) $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ 2 functors
3) $FG \xrightarrow{c} id_{\mathcal{D}}$, $id_{\mathcal{C}} \xrightarrow{u} GF$
4) st $F \to F(GF) = (FG)F \to F$, composite id
$G \to (GF)G = G(FG) \to G$, composite id_G

thm if there is a $\otimes$ invertible object $L \in \mathcal{D}(X)$,
and maps, $id_X^{LZ_D} \to L * 1_X = p_2^* L$
$= \Delta_! 1_X$
$f_! L = 1_X * L \to id_Y^{LZ_D} = 1_Y$
st $1_X \to 1_X * (L * 1_X) = (1_X * L) * 1_X \to 1_X$
and $L \to (L * 1_X) * L = L * (1_X * L) \to L$ are identities.
then f is cohomologically smooth. (verify Pduality)

Rk the converse is also true (it is an optimal condition)

proof: • why $L = f^!(1_Y)$ → automatic and tensor invertible is supposed.
• base change: base change along $Y' \to Y$, $X \xrightarrow{f} Y$, functor $LZ_D \to LZ_{D|\mathcal{C}/Y'}$ preserves adjunction

II. the $(\infty,2)$-variant of $\mathrm{Corr}(\mathcal{C},E)$

$\to$ homotopy cat: the 2-cat. LZ_D

how D is essentially (contain) the Fourier transform: unexpected

lemma in $\mathrm{Corr}(\mathcal{C},\ )$, all objects are self dual

idea if $E = $ all, all arrows exist in 2 directions in correspondences $\to$ symmetric

corollary: $\mathrm{Corr}(\mathcal{C},\ )$ is self-enriched (closed sym. mon.)

and $\mathrm{Hom}_{\mathrm{Corr}}(X,X') = X^\vee \otimes X' = X \otimes X' = X \times X'$

proof of the lemma. we want $X^\vee \otimes X \to *$ eval map.

$* \to X^\vee \otimes X$ co-eval map.

So we want $X \times X \to *$

$* \to X \times X$

$\Rightarrow$ take

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X \times X & & * \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & \\ * & & X \times X \end{tikzcd}$$

diagonal is in E!

<u>Reminder</u> $LZ_D \longrightarrow \mathrm{Cat}$

$X \longmapsto \mathcal{D}(X)$

$K \in D(X \times X') = \mathrm{Hom}_{LZ_D}(X,X') \longmapsto \phi_K : \mathcal{D}(X) \to \mathcal{D}(Y)$

very similar to $D: \mathrm{Corr}(\mathcal{C},E) \longrightarrow \mathrm{Cat}_\infty$.

except that now $\mathrm{Hom}(X,Y) = D(X \times X') = D(\underline{\mathrm{Hom}}_{\mathrm{Corr}}(X,X'))$

definition $\mathrm{Corr}(\mathcal{C})^D$: same as $\mathrm{Corr}(\mathcal{C})$

but the Hom is transfered along D lax monoidal

$(\mathrm{Hom}_{\mathrm{Corr}^D}(X,X') = \overline{D}(X \times X'))$

now $\mathrm{Corr}(\mathcal{C})^D$ is enriched in $\mathrm{Cat}_\infty \Rightarrow$ it is an $(\infty,2)$ category!

thm/def

$LZ_D \to Cat$ is the $(1,2)$-cat devd of the $(\infty,2)$ functor $\tilde{D} : Corr(\mathcal{C})^D \to Cat_\infty$

proof check that Fourier transform appears.

$$\begin{array}{ccc} Corr(\mathcal{C})^D & \longrightarrow & Cat_\infty \\ X & \longmapsto & D(X) \\ K \in Hom_{Corr^D}(X,Z) & \longmapsto & \phi_K : D(X) \to D(Z) \\ = D(X^\vee \times Z) & & \end{array}$$

$X^\vee = X$

how does $K \in D(X^\vee \times Z)$ acts on $D(X)$

there is $(X^\vee \times Z) \times X \xrightarrow{eval} Z$ induced by $X^\vee \times X \to 1$

K acts with $D(eval)$

$$\begin{tikzcd} & X \times Z \arrow[dl, "\Delta \times id"'] \arrow[dr, "p_2"] & \\ (X \times X) \times Z & & Z \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X \times X & & * \end{tikzcd}$$

lemma. the correspondence $f, g : X \to Z$

$$\begin{tikzcd} & X \arrow[dl, "(id{,}f)"'] \arrow[dr, "g"] & \\ X \times Z & & Y \end{tikzcd} \overset{D}{\rightsquigarrow} \begin{array}{c} D(X) \times D(Z) \longrightarrow D(Z) \\ (A,B) \longmapsto g_!(A \otimes f^*B) \end{array}$$

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Z & & Y \end{tikzcd} \overset{D}{\rightsquigarrow} g_! f^*$$

$$D\left(\begin{tikzcd} & X \times Z \arrow[dl, "p_1 \times id"'] \arrow[dr, "p_2"] & \\ X \times (X \times Z) & & Z \end{tikzcd} \right) \rightsquigarrow \begin{array}{c} D(X \times Z) \times D(X) \longrightarrow D(Z) \\ (K, B) \longmapsto p_{2!}(K \otimes p_1^* B) \end{array}$$

III. Example: Top.

smooth maps are cohom. smooth

$\rightarrow$ stability by composite, base change, open immersion

$\Rightarrow$ enough to do it for $f : \mathbb{R} \to *$!

$L = \mathbb{Z}[1]$ now exhibit maps $\alpha : \Delta_! \mathbb{Z} \longrightarrow \mathbb{Z}[1]$ (1)

$\beta : f_! L \to \mathbb{Z}$ (2)

$f_! L = \Gamma_c(\mathbb{R}, \mathbb{Z}[1])$

(2) in fact $\Gamma_c(\mathbb{R}, \mathbb{Z}) \simeq \mathbb{Z}[-1]$

$= \operatorname{colim}_{[-r,r] \subset \mathbb{R}} \left(\operatorname{fib} \left(\underset{\mathbb{Z}}{R\Gamma(\mathbb{R}, \mathbb{Z})} \longrightarrow \underset{\mathbb{Z} \oplus \mathbb{Z}}{R\Gamma(\mathbb{R} \setminus [-r,r], \mathbb{Z})} \right) \right)$

$= \operatorname{fib}(\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z})$

$= \mathbb{Z}[-1]$

(1) $j_! \mathbb{Z} \to \mathbb{Z} \to \Delta_* \mathbb{Z}$, with $\Delta_* = \Delta_!$

$R\Gamma(\mathbb{R}^2, \mathbb{Z}) \to R\Gamma(\mathbb{R}^2 \setminus \Delta, \mathbb{Z}) \to \operatorname{Hom}(\Delta_! \mathbb{Z}[-1], \mathbb{Z})$

$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$

then the compo are isos $\overset{\text{not id}}{\Longrightarrow}$ given or adj up to changing α.