ÉTALE COHOMOLOGY, PURITY AND FORMALITY WITH TORSION COEFFICIENTS

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Abstract. We use Galois group actions on étale cohomology to prove results of formality for dg-operads and dg-algebras with torsion coefficients. Our theory applies, among other related constructions, to the dg-operad of singular chains on the operad of little disks and to the dg-algebra of singular cochains on the configuration space of points in the complex space. The formality that we obtain is only up to a certain degree, which depends on the cardinality of the field of coefficients.

Contents

1. Introduction

The goal of this paper is to use the weight filtration in the étale cohomology of algebraic varieties in order to prove some formality results. Let us recall that an algebraic structure A in the category of chain complexes (like a dg-algebra, a commutative dg-algebra, a dg-operad, a dg-Hopf algebra, etc.) is said to be *formal* if A is connected to $H_*(A)$ by a zig-zag of quasi-isomorphisms that are compatible with the algebraic structure. This notion was first introduced in the setting of rational homotopy, in which a topological space X is said to be *formal* if its Sullivan's algebra of polynomial forms $\mathcal{A}_{pl}(X)$ is connected to its cohomology $H^*(X;\mathbb{Q})$ by a string of quasi-isomorphisms of commutative dg-algebras over $\mathbb{Q}$. In that case, any invariant of the rational homotopy type of X can be computed from the cohomology algebra of X.

This notion is extended to coefficients in an arbitrary commutative ring R, by asking that the complex of singular cochains $C^*(X,R)$ is quasi-isomorphic to its cohomology as dg-algebras. We point out here that in general the singular cochains do not have the structure

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of a commutative dg-algebra so the question of whether cochains and cohomology are quasi-isomorphic as commutative dg-algebras does not even make sense. On the other hand, the singular cochains have the structure of an E_∞-algebras and this object is a very strong invariant of the homotopy type of the space by a result of Mandell [Man06]. Therefore, one could also ask whether $C^*(X, R)$ and $H^*(X, R)$ are quasi-isomorphic as E_∞-algebra. However, this notion of formality almost never happens if R is not a $\mathbb{Q}$-algebra (for instance if $R = \mathbb{F}_p$, this implies that the Steenrod operations are trivial). Even though it is not the case that the homotopy type of the dg-algebra $C^*(X, R)$ is a complete invariant of the R-homotopy type of X, several invariants of X can be computed from it, such as the homology of ΩX with its Pontryagin products structure or the string topology of X when it is an orientable manifold. Let us also point out that if R is a field of characteristic zero, then the two notions of formality are equivalent [Sal17].

On the other hand, the theory of weights on the cohomology of algebraic varieties rests on fundamental ideas of Grothendieck and Deligne, and is strongly influenced by Grothendieck's philosophy of mixed motives. Even though a category of motives satisfying all the desired properties is still out of reach, the theory of weights is well-understood on the rational cohomology of complex algebraic varieties and on the étale cohomology of varieties over finite fields. In the case of complex varieties, the cohomology groups support a mixed Hodge structure which is pure for smooth projective varieties. Likewise, the étale cohomology groups of a variety over a finite field are acted on by the absolute Galois group of the field. Deligne [Del80] showed that the eigenvalues of the Frobenius are in general Weil numbers and that for smooth and projective varieties, the eigenvalues of the Frobenius action on the n-th cohomology groups are Weil numbers of pure weight n. Thus in general we indeed have a filtration of the étale cohomology groups which is pure for smooth projective varieties. Let us mention at this point that the Frobenius also acts on the étale cohomology groups with torsion coefficients which is key to the applications we have in mind. It is also important to note that the purity property holds more generally than just for smooth projective varieties: there are many interesting examples of singular and open varieties whose weights in cohomology turn out to be pure in a more flexible way which we call *α-purity*. For instance, the n-th cohomology group of the space of configurations of m points in the complex line has a Hodge structure that is pure of weight $2n$ (instead of n if it were a smooth projective variety).

The idea that purity implies formality goes back to Deligne, Griffiths, Morgan and Sullivan, who used the Hodge decomposition to show that compact Kähler manifolds are formal over $\mathbb{Q}$ [DGMS75]. Since then, Hodge theory has been used successfully several times to prove formality results over $\mathbb{Q}$ in different settings (see for instance [Mor78], [GNPR05], [Dup16], [Pet14], [CH17]). Using the methods of étale cohomology, Deligne [Del80] gave an alternative proof of formality for the $\mathbb{Q}_\ell$-homotopy type of smooth and proper complex schemes. However, it seems that the full power of Galois actions has not been further exploited to address formality questions especially in the case of torsion coefficients. Note that while Hodge theory is confined to rational coefficients, étale cohomology with torsion coefficients is perfectly well-defined and possesses interesting Galois group actions, making it a very valuable tool to study formality.

The notion of formality makes sense (and has proven to be very useful) in many other algebraic context outside dg-algebras, such as operads, operad algebras and symmetric monoidal functors. In this paper, we use the theory of weights in étale cohomology to prove partial results of the type "purity implies formality" when such algebraic structures arise from the category of algebraic varieties. Although the methods and conditions become quite technical, the theory has applications to very classical and well-known objects, such as the operad of little disks.

Let us briefly explain how purity can be used to prove formality on a simple example. Consider the étale cohomology of $X = \mathbb{P}^n_k$ where $k = \mathbb{F}_q$ is a finite field. In that case, the vector space $H^i_{et}(X_{\overline{k}}, \mathbb{Q}_\ell)$ is of dimension 1 if $i \leq 2n$ is even and of dimension zero otherwise. Moreover the Frobenius of k acts by multiplication by q^i on the $2i$-th cohomology group. Let us write $A = C^*_{et}(X, \mathbb{Q}_\ell)$ the dg-algebra of étale cochains on X. The Frobenius also acts on A and we can thus consider the subalgebra B where we only keep the generalized eigenspaces of the Frobenius for the eigenvalues that are powers of q. This dg-algebra is quasi-isomorphic to A as the other eigenspaces will not contribute to the cohomology. We can then split B as $B = \oplus_{i\in\mathbb{Z}} B(i)$ where $B(i)$ is the generalized eigenspace for the eigenvalue q^i. Observe that the cohomology of $B(i)$ will be concentrated in degree $2i$. Moreover, this decomposition is compatible with the multiplication. We have thus produced a multiplicative splitting of the Postnikov filtration of B which is another way to phrase formality. There is a technical difficulty with this sketch as in general, the dg-algebra A will not be finite dimensional and it does not make sense to decompose it as a sum of generalized eigenspaces. However, this issue can be fixed.

If we wanted instead to prove formality of $A = C^*_{et}(X, \mathbb{F}_\ell)$, the above sketch would also work modulo the fact that q^i can be equal to 1 in $\mathbb{F}_\ell$, since $\mathbb{F}_\ell^\times$ is a finite group. Therefore, we have a decomposition $B = \oplus_{i\in\mathbb{Z}/h} B(i)$ where h is the order of q in $\mathbb{F}_\ell$. This splitting is insufficient to prove formality in full generality but will imply formality when $h \geq n$.

We now spell out in more detail the results that we prove in this paper. Let K be a p-adic field and $\overline{K}$ its algebraic closure. We assume that $\overline{K}$ is embedded in $\mathbb{C}$. We will denote by Sch_K the category of schemes over K that are separated and of finite type. For $X \in \mathrm{Sch}_K$, the Galois action on étale cohomology actually exists at the cochain level and there exists a functorial dg-algebra $C^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ endowed with an endomorphism φ corresponding to the choice of a Frobenius lift. The dg-algebra of *étale cochains* relates to singular cochains as follows. Denote by X_{an} the complex analytic space underlying $X_\mathbb{C} = X \times_K \mathbb{C}$. Then we have quasi-isomorphisms

$$C^*_{sing}(X_{an}, \mathbb{F}_\ell) \longleftarrow C^*_{et}(X_\mathbb{C}, \mathbb{F}_\ell) \longrightarrow C^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell),$$

giving symmetric monoidal natural transformations of functors.

Let $k = \mathbb{F}_q$ be the residue field of K and denote by h the order of q in $\mathbb{F}_\ell^\times$. Let α be a positive rational number, with $\alpha < h$. We say that $H^n_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is a *pure Tate module of weight* αn if the only eigenvalue of the Frobenius is $q^{\alpha n}$, with $\alpha n \in \mathbb{N}$. If $\alpha n \notin \mathbb{N}$ we impose that $H^n_{et}(X_{\overline{K}}, \mathbb{F}_\ell) = 0$. If this condition is satisfied for every n, we say that X is *α-pure*.

To study the homologically graded case, we need to dualize the functor $C^*_{et}(-, \mathbb{F}_\ell)$. As a result, we obtain a lax symmetric monoidal ∞-functor $C^{et}_*(-, \mathbb{F}_\ell)$ of *étale chains* from $\mathbf{N}(\mathrm{Sch}_K)$ to the ∞-category of chain complexes equipped with an automorphism. This allows us to prove the following result:

Theorem 6.5. *Let P be a homotopically sound operad and let X be a P-algebra in* Sch_K*. Assume that for each color c of P, the cohomology $H^*_{et}(X(c)_{\overline{K}}, \mathbb{F}_\ell)$ is a pure Tate module of weight αn. Then $C_*(X_{an}, \mathbb{F}_\ell)$ is N-formal as a dg-P-algebra, with $N = \lfloor (h-1)/\alpha \rfloor$.*

By definition, *N-formality* means that we have maps of algebras relating the algebra and its cohomology, which induce isomorphisms in homology only up to degree N. The condition that the operad P is homotopically sound is a technical condition that is only needed in order to transfer N-formality from the ∞-category of P-algebras to the category of P-algebras.

We mention a few examples where this theorem applies. Denote by $\overline{\mathcal{M}}_{0,n}$ the moduli space of stable algebraic curves of genus 0 with n marked points. The operations that relate the different moduli spaces $\overline{\mathcal{M}}_{0,n}$ identifying marked points form a cyclic operad $\overline{\mathcal{M}}_{0,\bullet}$ in the category of smooth proper schemes over $\mathbb{Z}$, whose cohomology is pure. We deduce that the cyclic dg-operad $C_*((\overline{\mathcal{M}}_{0,\bullet})_{an}, \mathbb{F}_\ell)$ is $2(\ell-2)$-formal (Theorem 6.6). This extends the formality of $\overline{\mathcal{M}}_{0,\bullet}$ over $\mathbb{Q}$ proved in [GNPR05] to formality with torsion coefficients.

Using the action of the Grothendieck-Teichmüller group, we can also apply this machinery to the little disks operad $\mathcal{D}$ even though this operad is not an operad in the category of schemes. We prove $(\ell-2)$-formality for the dg-operad $C_*(\mathcal{D}, \mathbb{F}_\ell)$. A similar result applies to the framed little disks (Theorems 6.7 and 6.11). A direct consequence is the formality of the $(\ell-1)$-truncated operad $C_*(\mathcal{D}_{\leq(\ell-1)}, \mathbb{F}_\ell)$, where $\mathcal{D}_{\leq n}$ denotes the truncation of $\mathcal{D}$ in arity less than or equal to n. This result is optimal, since $C_*(\mathcal{D}_{\leq l}, \mathbb{F}_\ell)$ can be seen to be not formal (see Remark 6.9). This result potentially opens the door for a version of Kontsevich formality over a field of positive characteristic. The above method will be applied to higher dimensional little disks operads in forthcoming work of the second author with Pedro Boavida de Brito. The only extra ingredient needed is the construction of a Frobenius automorphism on the chains over those operads. We point out that Beilinson, in a letter to Kontsevich was conjecturing that the weight filtration in the triangulated category of motives could be used in order to prove formality of the little disks operad over the integers. Remark 6.9 shows that this was overly optimistic. However, we believe that our result is the closest one can get to answering Beilinson's question.

When working with coefficients in $\mathbb{Q}_\ell$ instead of $\mathbb{F}_\ell$ we are able to obtain more general and functorial formality results. Denote by $\mathrm{Sch}_K^{\alpha\text{-}pure}$ the symmetric monoidal category of schemes defined over K such that a Frobenius lift acts on the cohomology group $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ with eigenvalues that are Weil numbers of weight αn. Note that for $\alpha = 1$, the category $\mathrm{Sch}_K^{\alpha\text{-}pure}$ contains all schemes defined over K and admitting a smooth and proper model lift to $\mathcal{O}_K$ the ring of integers of K. We have:

Theorem 6.2. *The functor*

$$C_*^{sing}((-)_{an}, \mathbb{Q}_\ell) : \mathrm{Sch}_K^{\alpha\text{-}pure} \longrightarrow \mathrm{Ch}_*(\mathbb{Q}_\ell)$$

is formal as a lax symmetric monoidal functor.

As a consequence, any algebraic object in $\mathrm{Sch}_K^{\alpha\text{-}pure}$ (such as monoids, cyclic and modular operads, or any algebraic structure encoded by a colored operad) is sent to a formal object in $\mathrm{Ch}_*(\mathbb{Q}_\ell)$. This result is the étale counterpart to our previous results [CH17] on the formality of functors over $\mathbb{Q}$ using mixed Hodge theory. Its cochain version extends Deligne's formality of the $\mathbb{Q}_\ell$-homotopy for smooth proper schemes, to schemes in $\mathrm{Sch}_K^{\alpha\text{-}pure}$. We mention that this theorem answers a question asked by Petersen in the last remark of [Pet14].

Theorem 6.5 is restricted to non-negatively graded homological algebras. For cohomological dg-algebras we use the theory of free models to prove the following:

Theorem 8.2. *Let $X \in \mathrm{Sch}_K$ be a scheme over K. Assume that for all n, $H^n_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is a pure Tate module of weight αn. Then the following is satisfied:*

(i) *If $\alpha(k-2)/h$ is not an integer then all k-tuple Massey products vanish in $H^*(X_{an}, \mathbb{F}_\ell)$.*
(ii) *If $H^i(X_{an}, \mathbb{F}_\ell) = 0$ for all $0 < i \leq r$ then $H^n(X_{an}, \mathbb{F}_\ell)$ contains no non-trivial Massey products for all $n \leq \lceil \frac{hr}{\alpha} + 2r + 1 \rceil$.*
(iii) *If the graded algebra $H^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is simply connected, then $C^*_{sing}(X_{an}, \mathbb{F}_\ell)$ is an N-formal dg-algebra, with $N = \lfloor (h-1)/\alpha \rfloor$.*

The above result applies, for instance, to codimension c subspace arrangements defined over K (see Theorem 8.11). In particular, it applies to configuration spaces of points in $\mathbb{A}^n$. In Theorem 8.13 we show, that for every $d > 1$, the dg-algebra $C^*_{sing}(F_m(\mathbb{C}^d), \mathbb{F}_\ell)$ is N-formal, with

$$N = \left\lfloor \frac{(\ell-2)(2d-1)}{d} \right\rfloor .$$

We also obtain vanishing conditions for Massey products with torsion coefficients for any $d \geq 1$. This partially answers a question raised at the end of [Sal18], asking about the degree of obstructions to formality over $\mathbb{F}_\ell$. More generally, El Haouari [EH92] showed that a finite simply connected CW-complex is formal over $\mathbb{Q}$ if and only if it is formal over $\mathbb{F}_\ell$ for all primes ℓ but a finite number. Since N-formality implies formality for sufficiently large N, our results specify for which primes ℓ we do have formality over $\mathbb{F}_\ell$.

Finally let us mention some other related work. Ekedahl [Eke86] proved that for any prime ℓ, there exists a smooth simply connected complex projective surface X with a non-zero Massey product in $H^3(X, \mathbb{F}_\ell)$, which is a well-known obstruction to formality. This shows that in general the question of formality with coefficients in a finite field is much more delicate than in the rational case. More recently, Matei [Mat06] showed that for any prime ℓ, there is a complement of hyperplane arrangements X in $\mathbb{C}^3$ with a non-zero triple Massey product in $H^2(X, \mathbb{F}_\ell)$. Note that Matei's examples have torsion-free pure cohomology. We refer the reader to Remark 8.12 for a discussion of this result in relation to our result. Also, in [Sal18], Salvatore initiated the study of formality over arbitrary rings for the configuration spaces $F_m(\mathbb{R}^d)$ of m points in $\mathbb{R}^d$. He showed that if $m \leq d$, then $F_m(\mathbb{R}^d)$ is formal over any ring, but that $F_m(\mathbb{R}^2)$ is not formal over $\mathbb{F}_2$ when $m \geq 4$. Note that in all of the above cases, the corresponding spaces are known to be formal over the rationals. Lastly, there is an obstruction theory via Hochschild cohomology developed in [BB17], which allows to deduce formality over $\mathbb{Z}$ from formality over $\mathbb{Q}$ in certain quite restricted situations with torsion-free Hochschild cohomology. For instance, this gives formality over $\mathbb{Z}$ for complex projective spaces.

Outline of the paper. Let us briefly summarize the structure of this paper. In Section 2 we collect the main definitions on weights, Weil and Tate modules. We also introduce Weil and Tate complexes as more flexible structures than complexes of Weil and Tate modules. In Section 3 we construct the functor of étale cochains from finite type schemes over K to cochain complexes equipped with automorphisms and compare it with the singular cochains via Artin's theorem. The next three sections deal with the homologically graded case. In Section 4 we prove a Beilinson-type theorem for the categories of Weil and Tate complexes in terms of equivalences of symmetric monoidal ∞-categories. In Section 5 we give a criterion of N-formality for symmetric monoidal functors from chain complexes endowed with a certain $(\mathbb{Z}/m\mathbb{Z})$-grading. We apply this criterion in Section 6, to prove our main results over $\mathbb{Q}_\ell$ and over $\mathbb{F}_\ell$ in their chain version. The last two sections deal with the cohomologically graded case. Section 7 is the analogue of Section 5 for the case of cohomological dg-algebras. Again, we give a criterion of formality for dg-algebras endowed with a $(\mathbb{Z}/m\mathbb{Z})$-graded weight decomposition. These results are applied in Section 8 to prove our main results in the cochain setting.

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Notation. Throughout this paper, we fix a prime number p. Denote by K a p-adic field (i.e a finite extension of $\mathbb{Q}_p$) and by $q = p^n$ the cardinality of its residue field. We also assume that a choice of an embedding of K in $\mathbb{C}$ has been made. It follows that there is a preferred choice of embedding of the algebraic closure of $\overline{K}$ in $\mathbb{C}$, namely we can define $\overline{K}$ as the set of complex numbers that are algebraic over K. We denote by ℓ a prime number $\ell \neq p$ and write h for the order of q in the group $\mathbb{F}_\ell^\times$. All schemes over K will be assumed to be separated and of finite type.

2. Weil and Tate modules

In this section we make precise the notion of weight that we will consider. We actually consider two different notions of weight: for Weil modules over $\mathbb{Q}_\ell$ and for Tate modules over $\mathbb{F}_\ell$.

Definition 2.1. We say that an algebraic number μ is a *q-Weil number* if for all embeddings $\iota : \overline{\mathbb{Q}} \to \mathbb{C}$ we have the identity $|\iota(\mu)| = q^{n/2}$ for some integer n.

The integer n appearing in the above definition is called the *weight* of the Weil number. The product of two Weil numbers is a Weil number and the weight is additive with respect to this operation.

Definition 2.2. Let V be a finite dimensional $\mathbb{Q}_\ell$-vector space and φ an automorphism of V. We say that the pair (V, φ) is a *q-Weil module* if the characteristic polynomial of φ is in $\mathbb{Q}[t]$ and all of its roots are Weil numbers. A q-Weil module is said to be *pure of weight n* if the roots of the characteristic polynomial of φ are Weil numbers of weight n.

Definition 2.3. Let V be a finite dimensional $\mathbb{F}_\ell$-vector space and φ an automorphism of V. We say that the pair (V, φ) is a *q-Tate module* if the eigenvalues of φ are powers of q. Let $n \in \mathbb{N}$. A q-Tate module is said to be *pure of weight n* if the only eigenvalue of φ is q^n.

Remark 2.4. The name Tate module comes from the fact that those objects are constructed from the Tate twists $\mathbb{F}_\ell(n)$ defined below. Tate modules in our sense have little to do with the Tate module of an abelian variety. We hope that this does not lead to any confusion.

In practice, we will drop the mention of q and just say Weil module and Tate module respectively. Denote by WMod (resp. TMod) the category of Weil (resp. Tate) modules. Morphisms in these categories are given by morphisms of modules that commute with the given automorphisms.

Example 2.5. For n any integer, we denote by $\mathbb{Q}_\ell(n)$ the one dimensional $\mathbb{Q}_\ell$-vector space equipped with the automorphism $q^n\mathrm{id}$. Similarly we denote by $\mathbb{F}_\ell(n)$ the one-dimensional $\mathbb{F}_\ell$-vector space equipped with the automorphism $q^n\mathrm{id}$. Then $\mathbb{Q}_\ell(n)$ is a Weil module of pure weight $2n$ and $\mathbb{F}_\ell(n)$ is a Tate module of pure weight n.

Remark 2.6. As shown in the previous example the two notions of weight differ by a factor of 2. We could have instead made the two conventions compatible but then the underlying graded vector space of a Tate module constructed in Proposition 2.9 would have been concentrated in even degrees.

Lemma 2.7. *The categories* WMod *and* TMod *are symmetric monoidal abelian categories.*

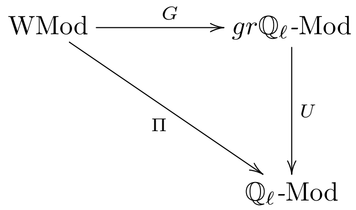
Proof. The symmetric monoidal structure is given by the obvious tensor product, noting that the product of Weil numbers is a Weil number and that product of powers of q are powers of q. In order to show that they are abelian, let $f : (V, \varphi) \to (V', \varphi')$ be a morphism of Weil (resp. Tate) modules. Then its kernel, cokernel, image and coimage are defined by taking the corresponding objects in $\mathbb{Q}_\ell$-modules (resp. $\mathbb{F}_\ell$-modules) together with the induced automorphisms. This gives WMod (resp. TMod) a pre-abelian structure. It now suffices to show that the canonical map $(\mathrm{Coim}(f),\varphi) \longrightarrow (\mathrm{Im}(f),\varphi')$ is an isomorphism of Weil (resp. Tate) modules. But this happens if and only if $\mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f)$ is an isomorphism of $\mathbb{Q}_\ell$-modules (resp. $\mathbb{F}_\ell$-modules). □

The following is a restatement in terms of symmetric monoidal functors of Deligne's weight grading (see [Del80, 5.3]). Consider the symmetric monoidal abelian category $gr\mathbb{Q}_\ell$-Mod of $\mathbb{Z}$-graded $\mathbb{Q}_\ell$-modules and denote by

$$U : gr\mathbb{Q}_\ell\text{-Mod} \longrightarrow \mathbb{Q}_\ell\text{-Mod}$$

the symmetric monoidal functor defined by forgetting degrees.

Lemma 2.8. *The functor* $\Pi : \mathrm{WMod} \longrightarrow \mathbb{Q}_\ell\text{-Mod}$ *defined by* $(V,\varphi) \mapsto V$ *admits a factorization*



into symmetric monoidal functors, where $G(V,\varphi)^n$ *is the direct sum of generalized eigenspaces corresponding to the eigenvalues that are Weil numbers of weight* n*.*

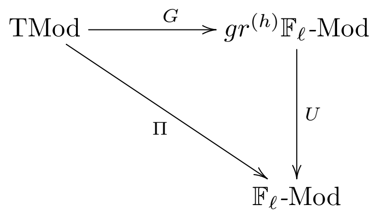
Proof. This is obvious modulo the fact that $G(V,\varphi)^n$ might a priori only exists when we extend scalars to $\overline{\mathbb{Q}}_\ell$. However observe that being a Weil number of weight n is invariant under the action of the Galois group of $\overline{\mathbb{Q}}_\ell$ over $\mathbb{Q}_\ell$. Since the characteristic polynomial of φ has coefficients in $\mathbb{Q}_\ell$, we deduce that this direct sum decomposition exists over $\mathbb{Q}_\ell$ as desired. □

For Tate modules, we use modules that are graded by a finite cyclic group instead of $\mathbb{Z}$. If $\mathcal{A}$ is a symmetric monoidal abelian category, we denote by $gr^{(m)}\mathcal{A}$ the category of $\mathbb{Z}/m$-graded objects of $\mathcal{A}$. It is a symmetric monoidal category, with the tensor product given by

$$(A \otimes B)^n := \sum_{a+b \equiv n \pmod m} A^a \otimes B^b.$$

The functor $U : gr^{(m)}\mathcal{A} \longrightarrow \mathcal{A}$ obtained by forgetting the degree is symmetric monoidal.

Lemma 2.9. *Let* h *be the order of* q *in* $(\mathbb{F}_\ell)^\times$*. The functor* $\Pi : \mathrm{TMod} \longrightarrow \mathbb{F}_\ell\text{-Mod}$ *defined by* $(V,\varphi) \mapsto V$ *admits a factorization*



into symmetric monoidal functors, where $G(V,\varphi)^n$ *is the generalized eigenspace for the eigenvalue* q^n*.*

We will consider more flexible structures than the ones provided by chain complexes of Weil and Tate modules.

Definition 2.10. A *Weil complex* is a pair (C,φ) where C is a chain complex of $\mathbb{Q}_\ell$-modules and φ is an endomorphism of C such that the pair $(H_n(C),H_n(\varphi))$ is a Weil module for all n. A *Tate complex* is a pair (C,φ) where C is a chain complex of $\mathbb{F}_\ell$-modules and φ is an endomorphism of C such that the pair $(H_n(C),H_n(\varphi))$ is a Tate module for all n.

Note that the homology of a Weil (resp. Tate) complex is always of finite type by assumption. We denote by $\mathrm{EComp}_{\mathbf{k}}$ the category of pairs (C,φ) where C is a chain complex over a field $\mathbf{k}$ and φ is an endomorphism. This category is symmetric monoidal with tensor product defined via the formula

$$(C,\varphi)\otimes(C',\varphi') := (C\otimes_{\mathbf{k}} C',\varphi\otimes_{\mathbf{k}}\varphi').$$

The category WComp of Weil complexes is a full sucbategory of $\mathrm{EComp}_{\mathbb{Q}_\ell}$ and similarly TComp, the category of Tate complexes is a full subcategory of $\mathrm{EComp}_{\mathbb{F}_\ell}$.

3. Functor of étale cochains

We denote by $\mathbf{k}$ the field $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$. In this section, we define the functor of étale cochains, from the category $\mathrm{Sch}_{\mathrm{K}}$ of schemes over K that are separated and of finite type, to the category $\mathrm{EComp}_{\mathbf{k}}$ of complexes with automorphisms.

Recall that a profinite set is a compact Hausdorff totally disconnected topological space. Alternatively, this is a pro-object in the category of finite sets. We denote by $\widehat{\mathrm{S}}$ the category of profinite spaces. Its objects are simplicial profinite sets or alternatively pro-objects in the category of simplicial sets that are degreewise finite and coskeletal [BHH17, Proposition 7.4.1].

Given a profinite set X and a finite commutative ring R, we denote by R^X, the ring of continuous maps $X\to R$ where R is given the discrete topology. For $R=\mathbb{Z}_\ell$, we denote by $\mathbb{Z}_\ell^X$ the ring of continuous functions with respect to the ℓ-adic topology on X defined by the formula

$$\mathbb{Z}_\ell^X := \lim_n (\mathbb{Z}/\ell^n)^X$$

Observe that this $\mathbb{Z}_\ell$-module is a submodule of the module of all functions $X\to\mathbb{Z}_\ell$ and is therefore a free $\mathbb{Z}_\ell$-module (since $\mathbb{Z}_\ell$ is a principal ideal domain). This implies that there is an isomorphism

$$(\mathbb{Z}_\ell^X)\otimes_{\mathbb{Z}_\ell}\mathbb{Z}/\ell^n \cong (\mathbb{Z}/\ell^n)^X.$$

Finally we denote by $\mathbb{Q}_\ell^X$ the ring $(\mathbb{Z}_\ell^X)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$.

If X is a profinite space and R is either $\mathbb{Z}/\ell^n$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, we denote by $S^\bullet(X,R)$ the cosimplicial commutative R-algebra given by R^{X_n} in degree n. We also denote by $C^*(X,R)$ the result of the application of the Dold-Kan equivalence to $S^\bullet(X,R)$. By the fact that the Dold-Kan construction is lax monoidal, we deduce that the resulting object is a dg-algebra. Note however, that it is not a commutative dg-algebra (instead it is naturally an algebra over the Barrat Eccles operad, see [BF04]).

Given a profinite space X, we denote by $H^*(X,R)$, the cohomology of $C^*(X,R)$ which is naturally a commutative graded algebra over R. By the universal coefficients theorem, we have a short exact sequence

$$0\to H^*(X,\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\mathbb{Z}/\ell^n\to H^*(X,\mathbb{Z}/\ell^n)\to \mathrm{Tor}(\mathbb{Z}/\ell^n,H^{*+1}(X,\mathbb{Z}_\ell))\to 0$$

and isomorphisms

$$H^*(X,\mathbb{Z}_\ell)\otimes\mathbb{Q}_\ell\cong H^*(X,\mathbb{Q}_\ell)$$

In this context, we also have a Künneth formula given by the following proposition.

Proposition 3.1. *Let X and Y be two profinite spaces. The canonical map*

$$S^\bullet(X,R)\otimes_R S^\bullet(Y,R)\to S^\bullet(X\times Y,R)$$

is an isomorphism, where R is one of the rings $\mathbb{Z}/\ell^n$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$.

Proof. It suffices to check it in each cosimplicial degree. Thus it is enough to prove that for X and Y two profinite *sets*, the canonical map

$$(R^X)\otimes_R(R^Y)\to R^{X\times Y}$$

is an isomorphism. When $R=\mathbb{Z}/\ell^n$, since the tensor product commutes with filtered colimits, we can assume that X and Y are finite sets in which case the statement is straightforward.

Now, we treat the case $R=\mathbb{Z}_\ell$. As we have said above the three modules $\mathbb{Z}_\ell^X$, $\mathbb{Z}_\ell^Y$ and $\mathbb{Z}_\ell^{X\times Y}$ are free. Therefore it suffices to prove that the canonical map is an isomorphism after tensoring with $\mathbb{Z}/\ell^n$ for each n which is what we have done in the previous paragraph.

Finally the case $R=\mathbb{Q}_\ell$ follows by tensoring with $\mathbb{Q}_\ell$ the isomorphism for $R=\mathbb{Z}_\ell$. □

Definition 3.2. We say that a map $X\to Y$ of profinite spaces is an *ℓ-complete equivalence* if the induced map

$$H^*(Y,\mathbb{Z}/\ell)\to H^*(X,\mathbb{Z}/\ell)$$

is an isomorphism.

Proposition 3.3. *Let $f:X\to Y$ be an ℓ-complete equivalence. Then:*

(1) *The induced map $H^*(Y,\mathbb{Z}/\ell^n)\to H^*(X,\mathbb{Z}/\ell^n)$ is an isomorphism for all n.*
(2) *The induced map $H^*(Y,\mathbb{Z}_\ell)\to H^*(X,\mathbb{Z}_\ell)$ is an isomorphism.*
(3) *The induced map $H^*(Y,\mathbb{Q}_\ell)\to H^*(X,\mathbb{Q}_\ell)$ is an isomorphism.*

Proof. We prove the first statement by induction on n. The case $n=1$ is the definition of an ℓ-complete equivalence. The short exact sequence

$$0\to\mathbb{Z}/\ell^n\to\mathbb{Z}/\ell^{n+1}\to\mathbb{Z}/\ell\to 0$$

induces a short exact sequence of cochain complexes

$$0\to C^*(X,\mathbb{Z}/\ell^n)\to C^*(X,\mathbb{Z}/\ell^{n+1})\to C^*(X,\mathbb{Z}/\ell)\to 0$$

since short exact sequences are stable under filtered colimits. It follows that we have a long exact sequence of cohomology groups and thus the statement for $n+1$ follows from the statement for n and the statement for 1.

In order to prove the second point, we observe that $C^*(X,\mathbb{Z}_\ell)$ is the limit of the diagram

$$n\mapsto C^*(X,\mathbb{Z}/\ell^n)$$

and all the transition maps in this diagram are epimorphisms. It follows that this limit is a homotopy limit.

Finally the third point follows from the universal coefficient theorem. □

Recall from Quick [Qui11] that there exists a functor Et from schemes over K to profinite spaces equipped with an action of the absolute Galois group of K. Recall that K is a p-adic field with residue field $\mathbb{F}_q$. It follows that there is a surjective map

$$\mathrm{Gal}(\overline{K}/K)\to\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q).$$

The target of this map is isomorphic to $\widehat{\mathbb{Z}}$ generated by the Frobenius. We make once and for all a choice of a lift of the Frobenius in $\mathrm{Gal}(\overline{K}/K)$ and this defines an inclusion

$$\mathbb{Z}\to\mathrm{Gal}(\overline{K}/K).$$

We can restrict the Galois action on $\mathrm{Et}(X)$ along the inclusion $\mathbb{Z} \to \mathrm{Gal}(\overline{K}/K)$ and we get a functor

$$\mathrm{Et} : \mathrm{Sch}_K \longrightarrow \widehat{\mathrm{S}}^{\mathbb{Z}}.$$

This functor has the property that there is an isomorphism

$$H^*(\mathrm{Et}(X), R) \cong H^*_{et}(X_{\overline{K}}, R)$$

where the $\mathbb{Z}$-action on the left-hand side corresponds to the action of the chosen Frobenius lift on the right-hand side.

Proposition 3.4. *Let X and Y be two K-schemes of finite type. The canonical map*

$$\mathrm{Et}(X \times Y) \to \mathrm{Et}(X) \times \mathrm{Et}(Y)$$

is an ℓ-complete equivalence.

Proof. In light of Proposition 3.1 and of the identification in the paragraph just above, we have to prove that the canonical map

$$H^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell) \otimes H^*_{et}(Y_{\overline{K}}, \mathbb{F}_\ell) \to H^*_{et}((X \times Y)_{\overline{K}}, \mathbb{F}_\ell)$$

is an isomorphism. This is just the Künneth theorem for étale cohomology [Del77, Cor. 1.11]. □

We now define a functor $C^*_{et}(-, \mathbf{k})$ on finite type schemes over K as the following composition

$$\mathrm{Sch}_K \xrightarrow{\mathrm{Et}} \widehat{\mathrm{S}}^{\mathbb{Z}} \xrightarrow{C^*(-,\mathbf{k})} (\mathrm{EComp}_{\mathbf{k}})^{\mathrm{op}}$$

Proposition 3.5. *This functor is oplax monoidal and the natural transformation*

$$C^*_{et}(X, \mathbf{k}) \otimes C^*_{et}(Y, \mathbf{k}) \to C^*_{et}(X \times Y, \mathbf{k})$$

is a quasi-isomorphism.

Proof. The first claim follows by observing that this functor is obtained by composing two oplax monoidal functors. The second claim follows by combining Proposition 3.4 and Proposition 3.1. □

We will need to compare the étale chain and cochains to singular chains and cochains. For X a scheme over K, we denote by X_{an} the complex analytic space underlying $X_{\mathbb{C}}$ (recall that $\overline{K}$ comes with a preferred embedding into $\mathbb{C}$).

There exists a zig-zag of symmetric monoidal natural transformation of functors from schemes over K to cochain complexes connecting $X \mapsto C^*_{sing}(X_{an}, \mathbf{k})$ and $X \mapsto C^*(X, \mathbf{k})$. The middle term in this zig-zag is the functor $X \mapsto C^*_{et}(X_{\mathbb{C}}, \mathbf{k})$. This is just the étale cochains of $X_{\mathbb{C}}$ and there is a map

$$C^*_{et}(X_{\mathbb{C}}, \mathbf{k}) \to C^*_{sing}(X_{an}, \mathbf{k})$$

This is the comparison between étale and singular cohomology which is a quasi-isomorphism for any scheme X (see [GAV73, XII, Theorem 4.1]). There is also a base change map

$$C^*_{et}(X_{\mathbb{C}}, \mathbf{k}) \to C^*_{et}(X_{\overline{K}}, \mathbf{k})$$

which is also a quasi-isomorphism for any scheme X (see [GAV73, XII, Corollary 5.4] for a proof in the proper case, see [Mil80, p. 231, Corollary 4.3] for the general case).

By Proposition 3.5, the functor $C^*_{et}(-, \mathbf{k})$ descends to a symmetric monoidal ∞-functor

$$C^*_{et}(-, \mathbf{k}) : \mathbf{N}(\mathrm{Sch}_k)^{\mathrm{op}} \longrightarrow \mathbf{EComp}_{\mathbf{k}}.$$

Since finite type schemes have finitely generated étale cohomology by [Del77, Corollary 1.10], by composing with the duality symmetric monoidal ∞-functor, we also get a covariant symmetric monoidal ∞-functor of étale chains

$$C_*^{et}(-,\mathbf{k}) : \mathbf{N}(\mathrm{Sch}_k) \longrightarrow \mathbf{EComp_k}.$$

4. A Beilinson-type Theorem for Weil and Tate complexes

Beilinson gave an equivalence of categories between the derived category of mixed Hodge structures and the category of mixed Hodge complexes. In this section, we prove analogous results for the categories of Weil and Tate complexes in terms of symmetric monoidal ∞-categories.

The symmetric monoidal structure on $\mathrm{EComp}_\mathbf{k}$ induces a symmetric monoidal ∞-category structure on $\mathbf{EComp_k}$ (the ∞-categorical localization of $\mathrm{EComp}_\mathbf{k}$ at quasi-isomorphisms. The purpose of this section is to give a simple model for the ∞-categories **TComp** and **WComp**. We first observe that there are canonical symmetric monoidal functors

$$\mathrm{Ch}_*(\mathrm{TMod}) \longrightarrow \mathrm{TComp} \text{ and } \mathrm{Ch}_*(\mathrm{WMod}) \longrightarrow \mathrm{WComp}.$$

Moreover, these two functors preserve quasi-isomorphisms on both sides. Therefore, they induce symmetric monoidal ∞-functors

$$\mathbf{Ch}_*(\mathrm{TMod}) \longrightarrow \mathbf{TComp} \text{ and } \mathbf{Ch}_*(\mathrm{WMod}) \longrightarrow \mathbf{WComp}.$$

We next describe the homotopy category $\mathrm{Ho}(\mathrm{EComp}_\mathbf{k})$ of chain complexes with endomorphisms defined by inverting quasi-isomorphisms, where $\mathbf{k}$ is a field. We follow the approach of [CG16] where the homotopy theory of diagrams of complexes is studied. Note that in loc.cit., complexes are assumed to be bounded. This hypothesis is not needed when working over a field, since every complex is homotopy equivalent to its homology.

Definition 4.1. Let (C,φ) and (C',φ') be objects in $\mathrm{EComp}_\mathbf{k}$. A *pre-morphism* (f,F) *of degree* n *from* C *to* C' is a pair of morphisms of graded $\mathbf{k}$-vector spaces $f : C[n] \to C'$ and $F : C[n-1] \to C'$.

Define the *differential* of such a pre-morphism by letting

$$D(f,F) := (df - (-1)^n fd, Fd + (-1)^n dF + (-1)^n(f\varphi - \varphi' f).$$

Definition 4.2. A *ho-morphism* $(f,F) : (C,\varphi) \to (C',\varphi')$ is a pre-morphism of degree 0 such that $D(f,F) = 0$. This is equivalent to having a diagram of morphisms of chain complexes

$$\begin{array}{ccc} C & \xrightarrow{\ \varphi\ } & C \\ {\scriptstyle f}\downarrow & \overset{F}{\searrow\!\!\searrow} & \downarrow{\scriptstyle f} \\ C' & \xrightarrow[\ \varphi'\ ]{} & C' \end{array}$$

which commutes up to a fixed homotopy F.

The composition of ho-morphisms is well-defined, given by

$$(f,F)\circ(g,G) := (fg, Fg + fG).$$

Moreover, it is straightforward to check that this composition is associative. We can thus define a category $\mathrm{EComp}_\mathbf{k}^h$ whose objects are those of $\mathrm{EComp}_\mathbf{k}$ and whose morphisms are given by ho-morphisms.

Definition 4.3. Let $(f, F), (g, G) : (C, \varphi) \to (C', \varphi')$ be two ho-morphisms. A *homotopy* from (f, F) to (g, G) a pre-morphism (h, H) of degree -1 such that $D(h, H) = (g-f, G-F)$. We then denote $(h, H) : (f, F) \simeq (G, g)$.

The homotopy relation between ho-morphisms is an equivalence relation compatible with composition (see [CG16, Lemma 3.8]).

Definition 4.4. Let $(f, F) : (C, \varphi) \to (C', \varphi')$ be a ho-morphism. The *mapping cylinder of* (F, f) is the object $(\mathcal{C}yl(f), \psi)$ of $\mathrm{EComp}_{\mathbf{k}}$ given by

$$\mathcal{C}yl(f)_n := C_{n-1} \oplus C'_n \oplus C_n$$

with the differential $D : \mathcal{C}yl(f)_n \to \mathcal{C}yl(f)_{n-1}$ and the endomorphism $\psi : \mathcal{C}yl(f) \to \mathcal{C}yl(f)$ given by

$$D := \begin{pmatrix} -d & 0 & 0 \\ -f & d & 0 \\ 1 & 0 & d \end{pmatrix} \quad \text{and} \quad \psi := \begin{pmatrix} \varphi & 0 & 0 \\ -F & \varphi' & 0 \\ 0 & 0 & \varphi \end{pmatrix}.$$

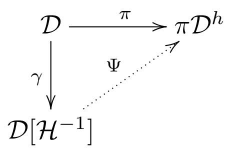
For any full subcategory $\mathcal{M} \subseteq \mathrm{EComp}_{\mathbf{k}}$, denote by $\pi\mathcal{M}^h$ the quotient category $\mathcal{M}^h/\simeq$ given by ho-morphisms modulo homotopy.

Theorem 4.5 (c.f. [CG16], Theorem 3.23)**.** *Let $\mathcal{D}$ be a full subcategory of* $\mathrm{EComp}_{\mathbf{k}}$ *and let $\mathcal{M} \subseteq \mathcal{D}$ be the full subcategory of objects with trivial differential. Assume that:*

(1) *The mapping cylinder of a ho-morphism between objects of $\mathcal{D}$ is an object of $\mathcal{D}$.*
(2) *For every object D of $\mathcal{D}$ there exists an object $M \in \mathcal{M}$ together with a ho-morphism from M to D which is a quasi-isomorphism.*

Then the inclusion induces an equivalence of categories $\pi\mathcal{M}^h \xrightarrow{\sim} \mathrm{Ho}(\mathcal{D})$.

Proof. We explain the main steps of the proof. Denote by $\mathcal{H}$ the class of morphisms of $\mathcal{D}$ which are homotopy equivalences in $\mathcal{D}^h$. Consider the solid diagram



where γ denotes the localization functor. By the universal property of the localized category, there is a unique dotted functor Ψ making the diagram commute. Since the mapping cylinder of ho-morphisms of $\mathcal{D}$ is an object of $\mathcal{D}$, this functor has an inverse (see [CG16, Theorem 3.18]). By restriction to the objects in $\mathcal{M}$, we obtain an equivalence of categories $\mathcal{D}[\mathcal{H}^{-1}]|_{\mathcal{M}} \xrightarrow{\sim} \pi\mathcal{M}^h$. Lastly, since every object has a model in $\mathcal{M}$, the inclusion induces an equivalence of categories of categories $\mathcal{D}[\mathcal{H}^{-1}]|_{\mathcal{M}} \xrightarrow{\sim} \mathcal{D}[\mathcal{Q}^{-1}]$ where $\mathcal{Q}$ denotes the class of quasi-isomorphisms. □

Lemma 4.6. *Let $\mathcal{D}$ be one of the categories* WComp, TComp, $\mathrm{Ch}_*(\mathrm{WMod})$ *or* $\mathrm{Ch}_*(\mathrm{TMod})$.

(1) *The mapping cylinder of a ho-morphism between objects of $\mathcal{D}$ is an object of $\mathcal{D}$.*
(2) *For any object (C, φ) in $\mathcal{D}$ there is a ho-morphism $(H_*(C), H_*(\varphi)) \to (C, \varphi)$ which is a quasi-isomorphism.*

Proof. The first condition follows from the fact that the characteristic polynomial P_ψ of the endomorphism ψ of Definition 4.4 is just P_φ^3 and that $H_*(\psi) \cong H_*(\varphi)$. Let us prove the second condition. Let (C, φ) be an object in $\mathcal{C}$. Choose a quasi-isomorphism $f : H_*(C) \to C$ by taking a section of the projection $Z_*(C) \to H_*(C)$. For any $x \in H_*(C)$, the difference

$(f \circ H_*(\varphi) - \varphi \circ f)x =: dw_x$ is a coboundary. The assignment $x \mapsto w_x$ defines a homotopy F in the diagram below This proves (2).

$$\begin{array}{ccc} H_*(C) & \xrightarrow{H_*(\varphi)} & H_*(C) \\ {\scriptstyle f}\downarrow & \overset{F}{\searrow\!\!\searrow} & \downarrow{\scriptstyle f} \\ C' & \xrightarrow{\varphi'} & C' \end{array} \; . \qquad \square$$

Theorem 4.7. *The symmetric monoidal ∞-functors*

$$\mathbf{Ch}_*(\mathrm{TMod}) \longrightarrow \mathbf{TComp} \text{ and } \mathbf{Ch}_*(\mathrm{WMod}) \longrightarrow \mathbf{WComp}$$

are equivalences of symmetric monoidal ∞-categories.

Proof. Note first that both ∞-categories $\mathbf{Ch}_*(\mathrm{WMod})$ and $\mathbf{WComp}$ are stable and that the inclusion functor is exact. Indeed, the stability of $\mathbf{WComp}$ follows from the observation that this ∞-category is the Verdier quotient at the acyclic complexes of the ∞-category of Weil complexes with the homotopy equivalences inverted. The latter ∞-category underlies a dg-category that can easily be seen to be stable. The stability of $\mathbf{Ch}_*(\mathrm{WMod})$ follows in a similar way. Since a complex of Weil modules is acyclic if and only if the underlying complex of $\mathbb{Q}_\ell$-modules is acyclic, the inclusion $\mathbf{Ch}_*(\mathrm{WMod}) \to \mathbf{WComp}$ is exact. Therefore, in order to prove that it is an equivalence of ∞-categories, it suffices to show that it induces an equivalence of homotopy categories $D(\mathrm{WMod}) \longrightarrow \mathrm{Ho}(\mathrm{WComp})$. By Lemma 4.6, Theorem 4.5 applies to both categories $\mathrm{Ch}_*(\mathrm{WMod})$ and WComp. This gives equivalences of categories

$$D(\mathrm{WMod}) = \mathrm{Ho}(\mathrm{Ch}_*(\mathrm{WMod})) \xleftarrow{\sim} \pi(gr\mathrm{WMod}^h) \xrightarrow{\sim} \mathrm{Ho}(\mathrm{WComp}).$$

The case of Tate complexes follows verbatim. $\square$

Remark 4.8. The above equivalences are also valid when we restrict to non-negatively (resp. non-positively) graded chain complexes.

5. Formality criteria for symmetric monoidal functors

Let $\mathcal{A}$ be a symmetric monoidal abelian category. In this section, we give a formality criterion for symmetric monoidal functors of chain complexes of objects of $\mathcal{A}$ equipped with an extra grading.

Definition 5.1. Let N be an integer. A morphism of chain complexes $f : A \to B \in \mathrm{Ch}_*(\mathcal{A})$ is called *N-quasi-isomorphism* if the induced morphism in homology $H_i(f) : H_i(A) \to H_i(B)$ is an isomorphism for all $i \leq N$.

Definition 5.2. Let $\mathcal{C}$ be a symmetric monoidal category and $F : \mathcal{C} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ a lax symmetric monoidal functor. Then F is said to be a *formal* (resp. *N-formal*) *lax symmetric monoidal* functor if there is a string of monoidal natural transformations of lax symmetric monoidal functors

$$F \overset{\Phi_1}{\Longleftarrow} F_1 \Longrightarrow \cdots \Longleftarrow F_n \overset{\Phi_n}{\Longrightarrow} H_* \circ F$$

such that for every object X of $\mathcal{C}$, the morphisms $\Phi_i(X)$ are quasi-isomorphisms (resp. N-quasi-isomorphisms).

Definition 5.3. Let $\mathcal{C}$ be a symmetric monoidal category and $F : \mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathcal{A})$ a lax symmetric monoidal functor (in the ∞-categorical sense). We say that F is a *formal* (resp. *N-formal*) *lax symmetric monoidal ∞-functor* if F and $H_* \circ F$ are quasi-isomorphic (resp. N-quasi-isomorphic) as lax monoidal functors from $\mathbf{N}(\mathcal{C})$ to $\mathbf{Ch}_*(\mathcal{A})$.

Remark 5.4. Denote by $t_{\leq N} : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the truncation functor defined by

$$(t_{\leq N}A)_n := \begin{cases} A_n & \text{if } n < N \\ A_N/\operatorname{Im}(d_{N+1}) & \text{if } n = N \\ 0 & \text{if } n > N \end{cases}.$$

This functor can be checked to be lax monoidal. Since it preserves quasi-isomorphisms, it descends to a lax monoidal ∞-functor $\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{Ch}_{\geq 0}(\mathcal{A})$. Moreover, for any lax monoidal functor $F : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the canonical map $F \to t_{\leq N}F$ is an N-quasi-isomorphism of lax monoidal functors. Likewise in the ∞-categorical case, given $F : \mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_{\geq 0}(\mathcal{A})$ a lax monoidal ∞-functor, the canonical map $F \to t_{\leq N}F$ is an N-quasi-iomorphism of lax monoidal ∞-functors. From this observation, we deduce that, in both cases, F is N-formal if and only if $t_{\leq N}F$ is formal.

Remark 5.5. The previous remark fails if we use the cohomological grading. In that case, we can define N-formality of an algebraic structure in cochain complexes as being a zig-zag of maps that induce isomorphisms in cohomology up to degree N. However, this notion cannot be interpreted as usual formality of the truncation since in the cohomological case, the truncation will not be lax monoidal (it is in fact oplax monoidal).

Every formal lax symmetric monoidal functor $\mathcal{C} \to \mathrm{Ch}_*(\mathcal{A})$ induces a formal lax symmetric monoidal ∞-functor $\mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathcal{A})$. We have the following partial converse:

Proposition 5.6 ([CH17], Corollary 2.4)**.** *Let $\mathbf{k}$ be a field of characteristic 0. Let $\mathcal{C}$ be a small symmetric monoidal category. Let $F : \mathcal{C} \to \mathrm{Ch}_*(\mathbf{k})$ be a lax symmetric monoidal functor. If $F : \mathbf{N}(\mathcal{C}) \longrightarrow \mathbf{Ch}_*(\mathbf{k})$ is formal as lax symmetric monoidal ∞-functor, then F is formal as lax symmetric monoidal functor.*

If the field $\mathbf{k}$ is not of characteristic zero, this proposition fails. The issue is that the homotopy theory of lax monoidal functor $\mathcal{C} \to \mathrm{Ch}_*(\mathbf{k})$ is usually not equivalent to the homotopy theory of lax monoidal ∞-functors $\mathbf{N}(\mathcal{C}) \to \mathbf{Ch}_*(\mathbf{k})$. We thus have to restrict to particularly well-behaved symmetric monoidal categories.

Definition 5.7. We say that a colored operad P is *admissible* if the category $\mathrm{Alg}_P(\mathrm{Ch}_*(\mathbf{k}))$ admits a model structure transferred along the forgetful functor

$$\mathrm{Alg}_P(\mathrm{Ch}_*(\mathbf{k})) \longrightarrow \mathrm{Ch}_*(\mathbf{k})^{\mathrm{Ob}(P)}.$$

We say that P is Σ-*cofibrant* if for any set $(a_1, \ldots, a_n; b)$ of colors of P, the symmetric group Σ_n acts freely on $P(a_1, \ldots, a_n; b)$. A colored operad P in sets is called *homotopically sound* if it is admissible and Σ-cofibrant.

Proposition 5.8. *Let $\mathbf{k}$ be a field of positive characteristic. Let P be a homotopically sound operad in sets. Let A be a P-algebra in $\mathrm{Ch}_*(\mathbf{k})$. If A is formal in $\mathbf{Alg}_P(\mathbf{Ch}_*(\mathbf{k}))$ (resp. N-formal in $\mathbf{Alg}_P(\mathbf{Ch}_{\geq 0}(\mathbf{k}))$), then A is formal in $\mathrm{Alg}_P(\mathrm{Ch}_*(\mathbf{k}))$ (resp. N-formal in $\mathrm{Alg}_P(\mathrm{Ch}_{\geq 0}(\mathbf{k}))$).*

Proof. Under those assumptions Hinich [Hin15] shows that the ∞-category underlying the model structure on $\mathrm{Alg}_P(\mathrm{Ch}_*(\mathbf{k}))$ for any field is equivalent to the ∞-category $\mathbf{Alg}_P(\mathbf{Ch}_*(\mathbf{k}))$. This immediately yields the result. The statement about N-formality follows from 5.4. □

Remark 5.9. The assumption that the operad is homotopically sound is not a siginficant restriction. Our paper gives a method for proving formality in the ∞-categorical sense. When the model structure on $\mathrm{Alg}_P(\mathrm{Ch}_*(\mathbf{k}))$ happens to model the ∞-category $\mathbf{Alg}_P(\mathbf{Ch}_*(\mathbf{k}))$, this implies formality in the model categorical sense. But in the other cases, the model category $\mathrm{Alg}_P(\mathrm{Ch}_*(\mathbf{k}))$ does not model the correct homotopy theory so the question of having formality is not particularly interesting.

Denote by $gr\mathcal{A}$ the category of graded objects of $\mathcal{A}$. It inherits a symmetric monoidal structure from that of $\mathcal{A}$, with the tensor product defined by

$$(A \otimes B)^n := \bigoplus_{a+b=n} A^a \otimes B^b.$$

The functor $U : gr\mathcal{A} \longrightarrow \mathcal{A}$ obtained by forgetting the degree is symmetric monoidal. The category of graded complexes $\mathrm{Ch}_*(gr\mathcal{A})$ inherits a symmetric monoidal structure via a graded Künneth morphism.

Definition 5.10. Given a rational number α, denote by $\mathrm{Ch}_*(gr\mathcal{A})^{\alpha\text{-pure}}$ the full subcategory of $\mathrm{Ch}_*(gr\mathcal{A})$ given by those graded complexes $A = \bigoplus A_n^p$ with *α-pure homology*:

$$H_n(A)^p = 0 \text{ for all } p \neq \alpha n.$$

Note that if $\alpha = a/b$, with a and b coprime, then the above condition implies that $H_*(A)$ is concentrated in degrees that are divisible by b, and in degree kb, it is pure of weight ka:

$$H_{kb}(A)^p = 0 \text{ for all } p \neq ka.$$

We have the following "purity implies formality" statement for symmetric monoidal functors.

Proposition 5.11 ([CH17], Proposition 2.7)**.** *Let α be a non-zero rational number. The functor $U : \mathrm{Ch}_*(gr\mathcal{A})^{\alpha\text{-pure}} \longrightarrow \mathrm{Ch}_*(\mathcal{A})$ defined by forgetting the degree is formal as a lax symmetric monoidal functor.*

To study formality in the torsion case, we will use a similar statement for chain complexes of $\mathbb{Z}/m$-graded objects. For the rest of this section we fix a positive integer m and a positive rational number α with $\alpha < m$.

Definition 5.12. Denote by $\mathrm{Ch}_*(gr^{(m)}\mathcal{A})^{\alpha\text{-pure}}$ the full subcategory of $\mathrm{Ch}_*(gr^{(m)}\mathcal{A})$ given by those $\mathbb{Z}/m$-graded complexes $A = \bigoplus A_n^p$ with *α-pure homology*:

$$H_n(A)^p = 0 \text{ for all } p \not\equiv \alpha n \pmod m.$$

Proposition 5.13. *Let $N = \lfloor (m-1)/\alpha \rfloor$. The functor*

$$U : \mathrm{Ch}_{\geq 0}(gr^{(m)}\mathcal{A})^{\alpha\text{-pure}} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$$

defined by forgetting the degree is N-formal as a lax symmetric monoidal functor.

Proof. We adapt the proof of [CH17, Proposition 2.7], to the $\mathbb{Z}/m$-graded setting.

Consider the truncation functor $\tau : \mathrm{Ch}_{\geq 0}(gr^{(m)}\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(gr^{(m)}\mathcal{A})$ defined by sending a $\mathbb{Z}/m$-graded chain complex $A = \bigoplus A_n^p$, to the graded complex given by:

$$(\tau A)_n^p := \begin{cases} A_n^p & n > \lceil p/\alpha \rceil \\ \mathrm{Ker}(d : A_n^p \to A_{n-1}^p) & n = \lceil p/\alpha \rceil \\ 0 & n < \lceil p/\alpha \rceil \end{cases}$$

for every $n \in \mathbb{Z}_{\geq 0}$ and every $0 \leq p < m$. Note that for each p, $\tau(A)_*^p$ is the chain complex given by the canonical truncation of A_*^p at $\lceil p/\alpha \rceil$, which satisfies

$$H_n(\tau(A)_*^p) \cong H_n(A_*^p) \text{ for all } n \geq \lceil p/\alpha \rceil.$$

Using the inequalities of the ceiling function $\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$ one easily verifies that τ is lax symmetric monoidal (see the proof of [CH17, Proposition 2.7]).

Consider the lax monoidal functor

$$t_{\leq N} H_* : \mathrm{Ch}_{\geq 0}(gr^{(m)}\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(gr^{(m)}\mathcal{A})$$

given by the N-truncated homology

$$t_{\leq N} H_n(A)^p := \left\{ \begin{array}{ll} H_n(A)^p & \text{if } n \leq N \\ 0 & \text{if } n > N \end{array} \right. .$$

Define a morphism $\Psi(A) : \tau A \longrightarrow t_{\leq N} H_*(A)$ by letting $\mathrm{Ker}(d) \twoheadrightarrow t_{\leq N} H_n(A)^p$ if $n = \lceil p/\alpha \rceil$ and $A_n^p \to 0$ if $n \neq \lceil p/\alpha \rceil$. We next show that this defines a monoidal natural transformation $\Psi : \tau \Rightarrow t_{\leq N} H_*$. It suffices to check that given $A, B \in \mathrm{Ch}_{\geq 0}(gr^{(m)} \mathcal{A})^{\alpha\text{-}pure}$, the diagram

$$\begin{array}{ccc} \tau A \otimes \tau B & \xrightarrow{\Psi(A)\otimes\Psi(B)} & t_{\leq N} H_*(A) \otimes t_{\leq N} H_*(B) \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \mu} \\ \tau(A \otimes B) & \xrightarrow{\Psi(A\otimes B)} & t_{\leq N} H_*(A \otimes B) \end{array}$$

commutes. The only non-trivial verification is for elements $a \in (\tau A)_p^n$ and $b \in (\tau B)_{p'}^{n'}$ with $n = \lceil p/\alpha \rceil$ and $n' = \lceil p'/\alpha \rceil$. Note that we have

$$n + n' = \lceil p/\alpha \rceil + \lceil p'/\alpha \rceil \geq \lceil (p + p')/\alpha \rceil.$$

We have the following three cases:

(1) If $p + p' < m$ and $n + n' = \lceil (p + p')/\alpha \rceil$ then $\Psi(a \otimes b) = [a \otimes b] \in t_{\leq N} H_{n+n'}(A \otimes B)$ and the diagram commutes.
(2) If $p + p' < m$ and $n + n' > \lceil (p + p')/\alpha \rceil$ then $\Psi(a \otimes b) = 0$. By α-purity we have $H_{n+n'}(A \otimes B)^{p+p'} = 0$, and the diagram trivially commutes.
(3) If $p + p' \geq m$ then $\Psi(a \otimes b) = 0$. Note that we have

$$n + n' \geq \lceil (p + p')/\alpha \rceil \geq \frac{p + p'}{\alpha} > \frac{m - 1}{\alpha} \geq N.$$

Therefore $t_{\leq N} H_{n+n'}(A \otimes B) = 0$ and the diagram trivially commutes.

The inclusion $\tau A \hookrightarrow A$ defines a monoidal natural transformation $\Phi : U \Rightarrow 1$. Also, there is an obvious monoidal natural transformation $\Upsilon : H_* \Rightarrow t_{\leq N} H_*$ defined by projection. All together, gives monoidal natural transformations

$$U \xLeftarrow{\Phi} U \circ \tau \xRightarrow{\Psi} U \circ t_{\leq N} H_* = t_{\leq N} H_* \circ U \xLeftarrow{\Upsilon} H_* \circ U.$$

It only remains to note that, if A has α-pure homology, then the morphism $\Phi(A)$ is a quasi-isomorphism and the morphisms $\Psi(A)$ and $\Upsilon(A)$ are N-quasi-isomorphisms. $\square$

Remark 5.14. The proof of the above proposition fails in the cohomological case. The main issue is that the functor τ is not lax symmetric monoidal in the $\mathbb{Z}/m$-graded situation. We will provide an alternative statement for cohomological dg-algebras in Section 7, using the theory of free models.

6. Main results in the homologically graded case

In this section, we use étale chains, together with the formality criterion of Section 5, to prove our main results of formality in their chain version.

6.1. **In the rational case.** Let α be a positive rational number. We define $\mathbf{WComp}^{\alpha\text{-}pure}$ to be the category of *α-pure Weil complexes*. Its objects are Weil complexes (C,φ) such that for all $n \geq 0$, the Weil module $(H_n(C), H_n(\varphi))$ is pure of weight αn. By convention, if $w \notin \mathbb{Z}$, a Weil module of weight w is the zero module.

Proposition 6.1. *The symmetric monoidal ∞-functor*

$$\mathbf{WComp}^{\alpha\text{-}pure} \longrightarrow \mathbf{Ch}_*(\mathbb{Q}_\ell)$$

is formal.

Proof. By Theorem 4.7 we have an equivalence of categories $\mathbf{Ch}_*(\mathrm{WMod}) \longrightarrow \mathbf{WComp}$. This clearly restricts to α-pure complexes. We can pick a symmetric monoidal inverse to this equivalence and we can write this functor as the composite

$$\mathbf{WComp}^{\alpha\text{-}pure} \longrightarrow \mathbf{Ch}_*(\mathrm{WMod})^{\alpha\text{-}pure} \longrightarrow \mathbf{Ch}_*(gr\mathrm{Vect}_{\mathbb{Q}_\ell})^{\alpha\text{-}pure} \longrightarrow \mathbf{Ch}_*(\mathbb{Q}_\ell)$$

where we used the factorization of functors given in Lemma 2.8. Then the last ∞-functor is formal by Proposition 5.11. □

Before stating our main theorem in the rational case, let us fix some notation. Let $\mathrm{Sch}_K^{\alpha\text{-}pure}$ denote the category of schemes over K such that $C^*(X,\mathbb{Q}_\ell)$ is an α-pure Weil complex. By the Künneth formula for étale and singular cohomology, it is clear that $\mathrm{Sch}_K^{\alpha\text{-}pure}$ is a symmetric monoidal category under the cartesian product.

Theorem 6.2. *The functor*

$$C_*^{sing}((-)_{an},\mathbb{Q}_\ell) : \mathrm{Sch}_K^{\alpha\text{-}pure} \longrightarrow \mathrm{Ch}_*(\mathbb{Q}_\ell)$$

is formal as a lax symmetric monoidal functor.

Proof. By definition of the category $\mathrm{Sch}_K^{\alpha\text{-}pure}$ and Proposition 5.6, it suffices to prove that the ∞-functor

$$C_*^{et}(-,\mathbb{Q}_\ell) : \mathrm{Sch}_K^{\alpha\text{-}pure} \longrightarrow \mathrm{Ch}_*(\mathbb{Q}_\ell)$$

of étale chains is formal. Since this functor factors through

$$\mathbf{WComp}^{\alpha\text{-}pure} \longrightarrow \mathbf{Ch}_*(\mathbb{Q}_\ell)$$

this result follows from the previous proposition. □

Remark 6.3. For $\alpha = 1$, the category $\mathrm{Sch}_K^{\alpha\text{-}pure}$ contains all smooth and proper schemes X over K admitting a lift $\mathcal{X}$ over $\mathcal{O}_K$ that is smooth and proper (such an X is said to have *good reduction*). Indeed, in that situation, the smooth and proper base change theorem for étale cohomology gives for all n an isomorphism

$$H^n_{et}(\mathcal{X}_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell) \cong H^n_{et}(X_{\overline{K}},\mathbb{Q}_\ell)$$

where the action of the Frobenius lift on the right hand side coincides with the action of the Frobnius of $\overline{\mathbb{F}}_q$ on the left hand side. The eigenvalues of the Frobenius on the left hand side are Weil numbers of weight n (see [Del74]).

6.2. **Applications.** As a direct application, we can recover certain results from [GNPR05]. The cyclic operad $\overline{\mathcal{M}}_{0,\bullet}$ lives in the category of smooth and proper schemes over $\mathbb{Z}$, therefore the cyclic dg-operad $C_*((\overline{\mathcal{M}}_{0,\bullet+1})_{an},\mathbb{Q}_\ell)$ is formal. We can also deduce formality over $\mathbb{Q}$ by the descent of formality result of [GNPR05].

Similarly, in [CH17] we proved that the operads $\mathcal{A}s_{S^1}$ and $\mathcal{A}s_{S^1} \rtimes S^1$ of non-commutative (framed) little disks, are formal over $\mathbb{Q}$. Both of these operads are the complex analytic space underlying operads in smooth schemes defined over $\mathbb{Z}$. In each arity, these operads are a product of copies of $\mathbb{G}_m$. Moreover, these operads have 2-pure cohomology (again it suffices to check it for $\mathbb{G}_m$). It thus follows that these two operads live in $\mathrm{Sch}_{\mathbb{Q}_p}^{2\text{-}pure}$, therefore their homology is formal over $\mathbb{Q}$.

6.3. **In the torsion case.** Let α be a positive rational number with $\alpha < h$, where we recall that h denotes the order of q in $\mathbb{F}_\ell^\times$. We define $\mathbf{TComp}_{\geq 0}^{\alpha\text{-}pure}$ to be the category of α-pure non-negatively graded Tate complexes.

Proposition 6.4. *The forgetful functor*

$$\mathbf{TComp}_{\geq 0}^{\alpha\text{-}pure} \longrightarrow \mathbf{Ch}_{\geq 0}(\mathbb{F}_\ell)$$

is N-formal as a lax monoidal ∞-functor, with $N = \lfloor (h-1)/\alpha \rfloor$.

Proof. By Lemma 2.9, the forgetful functor $\mathbf{TComp} \longrightarrow \mathbf{Ch}_*(\mathbb{F}_\ell)$ can be factored as follows

$$\mathbf{TComp} \longrightarrow \mathbf{Ch}_*(\mathrm{TMod}) \longrightarrow \mathbf{Ch}_*(gr^{(h)}\mathrm{Vect}_{\mathbb{F}_\ell}) \longrightarrow \mathbf{Ch}_*(\mathbb{F}_\ell)$$

where the first map is an inverse to the equivalence constructed in Theorem 4.7. We can restrict all categories to α-pure non-negatively graded objects and then the last functor is N-formal by Proposition 5.13. □

Our main result in the torsion case now follows easily.

Theorem 6.5. *Let P be a homotopically sound operad and let X be a P-algebra in Sch_K. Assume that for each color c of P, the cohomology $H_{et}^*(X(c)_{\overline{K}}, \mathbb{F}_\ell)$ is a pure Tate module of weight αn. Then $C_*(X_{an}, \mathbb{F}_\ell)$ is N-formal as a dg-P-algebra, with $N = \lfloor (h-1)/\alpha \rfloor$.*

Proof. Under those assumptions, by the previous proposition, we know that $C_*^{et}(X, \mathbb{F}_\ell)$ is N-formal in the ∞-categorical sense. The result thus follows from Proposition 5.8. □

6.4. **Applications.** We can again consider the cyclic operad $\overline{\mathcal{M}}_{0,\bullet}$. As we have said before, this operad lives in the category of smooth and proper schemes over $\mathbb{Z}$.

Theorem 6.6. *The cyclic dg-operad $C_*((\overline{\mathcal{M}}_{0,\bullet})_{an}, \mathbb{F}_\ell)$ is $2(\ell-2)$-formal.*

Proof. It suffices to note that the cohomology $H_{et}^n((\overline{\mathcal{M}}_{0,n})_k, \mathbb{F}_\ell)$ is a Tate module that is pure of weight $n/2$. □

We also have a statement for the little disks operad $\mathcal{D}$. This operad does not quite fit our theorem since this is not an operad in the category of schemes. Nevertheless, one can construct a model $C_*(\mathcal{D}, \mathbb{F}_\ell)$ equipped with an action of the Grothendieck-Teichmüller group $\widehat{GT}$, which promotes $C_*(\mathcal{D}, \mathbb{F}_\ell)$ to an operad in the category of Tate complexes.

Theorem 6.7. *The dg-operad $C_*(\mathcal{D}, \mathbb{F}_\ell)$ is $(\ell-2)$-formal.*

Proof. There is an action of $\widehat{GT}$ on an operad $\widehat{B\mathcal{PAB}}$ constructed by Drinfeld in [Dri90] (see also [Hor17, Section 7] for more details). This is an operad in the category of simplicial profinite sets. We can apply $S^\bullet(-, \mathbb{F}_\ell)$ to this object (this construction is defined in Section 3) and we get a cosimplicial cooperad with an action of $\widehat{GT}$ that we can then dualize to get an operad in simplicial $\mathbb{F}_\ell$-vector spaces equipped with an action of $\widehat{GT}$. Finally we can apply the Dold-Kan construction to this operad to end up with a dg-operad. We claim that the resulting operad is quasi-isomorphic to $C_*(\mathcal{D}, \mathbb{F}_\ell)$. A similar statement is proven in [BdBHR17, Theorem 9.1].

The group $\widehat{GT}$ comes with a surjective map

$$\chi_\ell : \widehat{GT} \to \mathbb{Z}_\ell^\times$$

that factors the cyclotomic character of the absolute Galois group of $\mathbb{Q}$ (section 3.1 of [Sch97]). Moreover, the action of an element $\varphi \in \widehat{GT}$ on $H_k(\mathcal{D}(2), \mathbb{F}_\ell)$ is trivial in degree zero and given by $\chi_\ell(\varphi)\mathrm{id}$ in degree 1. This can be proved by using the explicit action of $\widehat{GT}$ on $B\widehat{PAB}(2) \cong B\widehat{\mathbb{Z}}$ where $\widehat{Z}$ denotes the profinite completion of the group $\mathbb{Z}$. As an operad $H_*(\mathcal{D}, \mathbb{F}_\ell)$ is generated by operations of arity 2. Therefore, we can deduce that the action of φ on $H_n(\mathcal{D}(n), \mathbb{F}_\ell)$ is given by multiplication by $\chi_\ell(\varphi)^n$ (Petersen uses the analogous argument in the rational case in [Pet14]). In particular, let p be a prime number such that p generates $\mathbb{F}_\ell^\times$ and let φ be an element of $\widehat{GT}$ such that $\chi_\ell(\varphi) = p$ (such an φ exists by surjectivity of χ_ℓ). Then from the previous discussion the pair $(C_*(\mathcal{D}, \mathbb{F}_\ell), \varphi)$ is an operad in the category of Tate complexes that is pure of weight 1. □

Let us denote by $\mathcal{D}_{\leq n}$ the truncation of the little disks operad in arity less than or equal to n (that is we only keep the composition maps that only involve those arities). Then $C_*(\mathcal{D}_{\leq n}, \mathbb{F}_\ell)$ is an n-truncated dg-operad.

Corollary 6.8. *The $(\ell-1)$-truncated operad $C_*(\mathcal{D}_{\leq(\ell-1)}, \mathbb{F}_\ell)$ is formal.*

Proof. By the previous theorem, we deduce that $C_*(\mathcal{D}_{\leq(\ell-1)}, \mathbb{F}_\ell)$ is $(\ell-2)$ formal. On the other hand the homology of $\mathcal{D}(n)$ is concentrated in degrees less than or equal to $n-1$. Therefore, the $\ell-2$-truncation map

$$C_*(\mathcal{D}_{\leq \ell-1}, \mathbb{F}_\ell) \to t_{\leq \ell-2} C_*(\mathcal{D}_{\leq \ell-1}, \mathbb{F}_\ell)$$

is a quasi-isomorphism. □

Remark 6.9. This corollary is in some sense optimal. Indeed $C_*(\mathcal{D}_{\leq l}, \mathbb{F}_\ell)$ is not formal. If it were the case, it would imply that there is a Σ_ℓ-equivariant quasi-isomorphism.

$$C_*(\mathcal{D}(\ell), \mathbb{F}_\ell) \simeq H_*(\mathcal{D}(\ell), \mathbb{F}_\ell)$$

This would mean that the homotopy orbit spectral sequence

$$H_*(\Sigma_\ell, H_*(\mathcal{D}(\ell), \mathbb{F}_\ell)) \implies H_*(\mathcal{D}(\ell)_{h\Sigma_\ell}, \mathbb{F}_\ell)$$

collapses at the E^2-page. However, this is not the case. Indeed the E^2-page has non-trivial homology in arbitrarily high degree. On the other hand, the space $\mathcal{D}(\ell)_{h\Sigma_\ell}$ has the homotopy type of the space of unordered configurations of ℓ points in the plane which is a manifold and in particular has bounded above homology.

Remark 6.10. This result will be generalized to higher dimensional little disks operads in forthcoming work of the second author with Pedro Boavida de Brito. The strategy of proof is exactly the same: we construct a Frobenius automorphism on the singular chains over the little n-disks operad which is such that the resulting operad in Tate complexes is pure.

We have a similar theorem for the framed little disks operad $\mathcal{FD}$.

Theorem 6.11. *The dg-operad $C_*(\mathcal{FD}, \mathbb{F}_\ell)$ is $(\ell-2)$-formal.*

Proof. The proof is entirely analogous once we have an action of $\widehat{GT}$ on the framed little disks operad. This is constructed in [BdBHR17, Theorem 8.4] and the effect of this action on homology is explained in [BdBHR17, Theorem 9.1]. □

7. Weight decompositions and formality of cohomological dg-algebras

In this section we give a criterion of formality for dg-algebras over an arbitrary field $\mathbf{k}$ equipped with a $(\mathbb{Z}/m\mathbb{Z})$-graded weight decomposition.

Definition 7.1. Let $N \geq 0$ be an integer. A dg-algebra A is said to be *N-formal* if there is a string of N-quasi-isomorphisms of dg-algebras from A to its cohomology $H^*(A)$, considered as a dg-algebra with trivial differential.

From now on, we fix a positive integer m and a positive rational number α with $\alpha < m$.

Definition 7.2. Let A be a non-negatively graded dg-algebra over $\mathbf{k}$ and m a positive integer. A *$gr^{(m)}$-weight decomposition* of A is a direct sum decomposition

$$A^n = \bigoplus_{p=0}^{m-1} A^n_p$$

of each vector space A^n, such that:

(1) $dA^n_p \subseteq A^{n+1}_p$ for all $n \geq 0$ and all $0 \leq p \leq m-1$.
(2) $A^n_p \cdot A^{n'}_{p'} \longrightarrow A^{n+n'}_{p+p' \,(\text{mod } m)}$ for all $n, n' \geq 0$ and all $0 \leq p, p' \leq m-1$.

Given $x \in A^n_p$ we will denote by $|x| = n$ its *degree* and by $w(x) = p$ its *weight*.

Denote by $gr^{(m)}\mathsf{DGA}_{\mathbf{k}}$ the category of dg-algebras with $gr^{(m)}$-weight decompositions. Note that these are just monoids in $\mathrm{Ch}^{\geq 0}(gr^{(m)}\mathbf{k})$.

Definition 7.3. The cohomology of a dg-algebra $A \in gr^{(m)}\mathsf{DGA}_{\mathbf{k}}$ is said to be *α-pure* if

$$H^n(A)_p = 0 \text{ for all } p \not\equiv \alpha n \,(\text{mod } m).$$

Denote by $gr^{(m)}\mathsf{DGA}^{\alpha\text{-}pure}_{\mathbf{k}}$ the category of dg-algebras with α-pure cohomology.

By obvious degree-weight reasons, the condition of α-purity has direct consequences on the vanishing of Massey products. We have:

Proposition 7.4. *Let $A \in gr^{(m)}\mathsf{DGA}^{\alpha\text{-}pure}_{\mathbf{k}}$ and $k \geq 3$ an integer. If $\frac{\alpha(k-2)}{m}$ is not an integer, then all k-tuple Massey products vanish in $H^*(A)$.*

Proof. Given cohomology classes $x_i \in H^{n_i}(A)_{w_i}$, with $1 \leq i \leq k$, a defining system for the k-tuple Massey product $\langle x_1, \cdots, x_k \rangle$ is given by elements $x_{i,j} \in A$ with $1 \leq i \leq j \leq k$ such that $(i,j) \neq (1,k)$ satisfying (modulo signs)

$$[x_{i,i}] = x_i \text{ and } dx_{i,j} = \sum_{q=i}^{j-1} x_{i,q} x_{q+1,j}.$$

Then $\langle x_1, \cdots, x_k \rangle$ is defined via the cohomology class of

$$x := \sum_{q=1}^{k-1} x_{1q} \cdot x_{q+1,k}$$

where again we omitted signs, since they are irrelevant for the proof. Let $n_i := |x_i|$ and $w(x_i)$ denote the degree and the weight of x_i respectively. Then from the above formulas we obtain

$$|x_{ij}| = \sum_{q=i}^{j} n_i + i - j \text{ and } w(x_{ij}) = \sum_{q=i}^{j} w_i.$$

Let $n := \sum_{i=1}^k n_i$ and $w := \sum_{i=1}^k w_i$. The expression for x gives $|x| = n-k+2$ and $w(x) = w$. Now, α-purity tells us that $w(x_i) \equiv \alpha n_i \,(\text{mod } m)$ and hence k-tuple Massey products live in

$H^{n-k+2}(A)_w$, where $w \equiv \alpha n \,(\mathrm{mod}\ m)$ and $n \geq 3$. Such cohomology groups are non-trivial only when $w \equiv \alpha(n-k+2) \,(\mathrm{mod}\ m)$. This gives the condition $\alpha(k-2) \equiv 0 \,(\mathrm{mod}\ m)$. □

For simply connected dg-algebras, the above proposition tells us that all higher Massey products living in sufficiently low-degree cohomology will vanish, regardless of their length.

Definition 7.5. A dg-algebra A is said to be *cohomologically connected* if if the unit map induces an isomorphism $H^0(A) \cong \mathbf{k}$. It is *simply connected* if, in addition, $H^1(A) = 0$. Let $r \geq 1$. Then A is called *r-connected* if, in addition, $H^i(A) = 0$ for all $1 \leq i \leq r$.

Proposition 7.6. *Let $A \in gr^{(m)}\mathsf{DGA}_{\mathbf{k}}^{\alpha\text{-}pure}$ be simply connected. Then $H^n(A)$ contains no non-trivial Massey products, for all $n \leq \lceil \frac{m}{\alpha} \rceil + 3$. More generally, if A is r-connected with $r > 0$ then $H^n(A)$ contains no non-trivial Massey products, for all $n \leq \lceil \frac{mr}{\alpha} \rceil + 2r + 1$.*

Proof. If $H^i(A) = 0$ for all $0 < i \leq r$, then a k-tuple Massey product will have degree at least $rk + 2$. The condition $rk + 2 \leq \lceil \frac{mr}{\alpha} \rceil + 2r + 1$ gives $\alpha(k-2) < m$. □

The following Lemma shows that for simply connected objects in $gr^{(m)}\mathsf{DGA}_{\mathbf{k}}^{\alpha\text{-}pure}$, one may construct a model whose weights are controlled in a certain way.

Lemma 7.7. *Let $A \in gr^{(m)}\mathsf{DGA}_{\mathbf{k}}^{\alpha\text{-}pure}$ be simply connected. Then there exists an N-quasi-isomorphism $f : M \to A$ in $\mathrm{Ch}^{\geq 0}(gr^{(m)}\mathbf{k})^{\alpha\text{-}pure}$ such that:*

1. *M is generated by elements of degree $\leq N$.*
2. *The differential on $M^n_{\alpha n}$ is trivial for all $\alpha n \leq m - 1$.*
3. *Let $\alpha n < p \leq m - 1$. If $M^n_p \neq 0$ then $p \leq \alpha(2n-2)$.*

Proof. We use the standard theory of free models for dg-algebras over an arbitrary field (see [HL88]). We will define, inductively over the degree $n \geq 0$, a sequence of dg-algebras $M\langle n\rangle$ with $gr^{(m)}$-weight decompositions $M\langle n\rangle^k = \bigoplus M\langle n\rangle^k_p$ and morphisms $f_n : M\langle n\rangle \to A$ compatible with weight decompositions and satisfying the following conditions:

(a_n) The dg-algebra $M\langle n\rangle$ is a free extension of $M\langle n-1\rangle$ by generators of degree n and weights $w = p \,(\mathrm{mod}\ m)$, with $\alpha n \leq p \leq \alpha(2n-2)$.

(b_n) The map $H^i(f_n)$ is an isomorphism for all $i \leq n$ and a monomorphism for $i = n+1$.

Then the morphism

$$f : \bigcup_n f_n : \bigcup_{n \leq N} M\langle n\rangle \to A$$

will be our model for A.

Let $M\langle 1\rangle = \mathbf{k}$ concentrated in weight 0 and degree 0 and define $f_1 : M\langle 1\rangle \to A$ to be the unit map. Conditions (a_1) and (b_1) are trivially satisfied. Assume inductively that we have defined $f_{n-1} : M\langle n-1\rangle \to A$ satisfying (a_{n-1}) and (b_{n-1}). for each $0 \leq p < m$, let $V_p := H^n(C(f_{n-1}))_p$ and consider it as a bigraded vector space of degree n and pure weight p. Define a differential $d : V_p \to M\langle n-1\rangle^{n+1}_p$ and a map $f_n : V_p \to A^n_p$ by taking a section of the projection

$$H^n(C(f_{n-1}))_p \twoheadleftarrow Z^n(C(f_{n-1}))_p \subset M\langle n-1\rangle^{n+1}_p \oplus A^n_p.$$

These define a differential on

$$M\langle n\rangle := M\langle n-1\rangle \sqcup T(V)$$

and a map $f_n : M\langle n\rangle \to A$ compatible with the weight decompositions. By classical arguments, since A is simply connected and $H^n(A) = \bigoplus H^n(A)_p$, condition ($b_n$) is satisfied. Let us prove (a_n). Note that elements in V_p arise either from the cokernel of $H^n(f)_p$, which by α-purity, is non-trivial only for $p = \alpha n$, or from elements in the kernel of $H^{n+1}(f)_p$. Since

$M^1 = 0$, elements in this kernel are represented by sums of products $x_1 \cdots x_k \in M\langle n-1\rangle^{n+1}$ with $k \geq 2$. Let $n_i := |x_i|$. By induction hypothesis we have $w(x_i) \equiv p_i \,(\text{mod } m)$ with $\alpha n_i \leq p_i \leq \alpha(2n_i - 2)$. We get

$$n+1 = n_1+\cdots+n_k \leq p_1+\cdots+p_k \leq (2n_1-2)\alpha+\cdots+(2n_k-2)\alpha \leq 2(n+1)\alpha-2k\alpha \leq (2n-2)\alpha.$$

Therefore the generators of V_p have degree n and weights $w \equiv p\,(\text{mod } m)$ with $\alpha n \leq p \leq (2n-2)\alpha$ and condition (a_n) is satisfied. This ends the inductive step.

We now prove that the differential on $M^n_{\alpha n}$ is trivial for all $\alpha n \leq m-1$. In fact, we will show that $M^{n+1}_{\alpha n} = 0$ for all $\alpha n \leq m-1$. Assume that $x \in M^{n+1}_{\alpha n}$. By construction, $w(x) = \alpha n = p - \lambda n$ with $\lambda \in \mathbb{Z}_{>0}$ and $\alpha(n+1) \leq p \leq 2n\alpha$. This gives the condition $\lambda m \leq \alpha n$, which contradicts the condition that $\alpha n \leq m-1$. Therefore $x = 0$. □

Proposition 7.8. *Let $A \in gr^{(m)}\mathsf{DGA}_{\mathbf{k}}^{\alpha\text{-pure}}$ be simply connected. Then it is N-formal, with $N = \lfloor (m-1)/\alpha \rfloor$.*

Proof. Let $f : M \to A$ be an N-model given by Lemma 7.7. It suffices to prove that M is N-formal. As a bigraded vector space, M may be decomposed into $M = A \oplus D \oplus B$ where D denotes the diagonal $p = \alpha n$ truncated up to degree $\alpha n \leq m-1$, and A and B are the direct sum of all vector spaces above and below the diagonal respectively: by letting

$$I_d := \{(n,p); p = \alpha n \leq m-1\} \text{ and } I_a := \{(n,p); n < \frac{m-1}{\alpha}, p > \alpha n\}$$

we may write

$$D = \bigoplus_{(n,p)\in I_d} M^n_p,\, A = \bigoplus_{(n,p)\in I_a} M^n_p \text{ and } B = \bigoplus_{(n,p)\notin I_d \cup I_a} M^n_p.$$

By degree-weight reasons, B is a dg-algebra ideal of M. By Lemma 7.7 the differential of D is trivial and we have $H^n(B) = 0$ for all $n \leq (m-1)/\alpha$. Therefore the morphism of dg-algebras $\pi : M \twoheadrightarrow M' := M/B$ induces an isomorphism in cohomology $H^n(\pi)$ for all $n \leq (m-1)/\alpha$. Consider the projection morphism $M' \to H^*(M')$ given by $A \mapsto 0$ and $D \mapsto D/\mathrm{Im}(d)$. To see that it is a quasi-isomorphism of dg-algebras, it suffices to see that $A \times M' \subseteq A$. Let $x \in A$ and $y \in M'$. By Lemma 7.7 we have

$$\begin{array}{lll} w(x) \equiv p_x \,(\text{mod } m) & \text{with} & \alpha|x| < p_x \leq (2|x|-2)\alpha, \\ w(y) \equiv p_y \,(\text{mod } m) & \text{with} & \alpha|y| \leq p_y \leq (2|y|-2)\alpha. \end{array}$$

Assume that $0 \neq x \cdot y \in D$ and let $n := |x| + |y|$. Then there is $\lambda \in \mathbb{Z}$ such that $p_x + p_y = \alpha n + \lambda m$. Since $\alpha n < p_x + p_y$ we have $\lambda_\in \mathbb{Z}_{>0}$. This gives $\alpha n + m \leq \alpha n + \lambda m \leq 2\alpha n - 4\alpha$ and hence $m \leq \alpha(n-4)$ which is a contradiction, since $n \leq m-1$.

□

8. Main results in the cohomologically graded case

Recall that objects in the category of Tate complexes $(\mathrm{TComp})^{\mathrm{op}}$ are cochain complexes over $\mathbb{F}_\ell$ enriched with automorphisms φ giving Tate modules in cohomology. We will show that every monoid in $(\mathrm{TComp})^{\mathrm{op}}$ is quasi-isomorphic to a dg-algebra in $gr^{(h)}\mathsf{DGA}_{\mathbb{F}_\ell}$, where we recall that h denotes the order of q in $\mathbb{F}_\ell^\times$. As a consequence, the formality criterion of Proposition 7.8 applies to étale cochains.

Proposition 8.1. *Let A be a cohomologically connected monoid in $(\mathrm{TComp})^{\mathrm{op}}$. Then there is a dg-algebra M in $gr^{(h)}\mathsf{DGA}_{\mathbb{F}_\ell}$ together with a quasi-isomorphism $M \to A$ of dg-algebras, such that $H^n(M)_k$ corresponds to the generalized eigenspace of $(H^n(A), \varphi)$ of the eigenvalue q^k. Moreover, if A is simply connected, we can pick M to be of finite type.*

Proof. A model for A may be built inductively by free extensions of the type

$$M \longrightarrow \tilde{M} := M \sqcup_d T(V)$$

where $d : V \to M$ is a linear map of degree 1. In our case, such extensions will carry an endomorphism as well as a compatible $gr^{(m)}$-weight decomposition as we next explain.

Assume that we have constructed a monoid (M, φ) in $(\mathrm{TComp})^{\mathrm{op}}$ together with a ho-morphism $(f, F) : M \to A$ in such a way that M has a $gr^{(h)}$-weight decomposition $M = \bigoplus M^n_k$ where M^n_k is the generalized eigenspace of M^n of of eigenvalue q^k with respect to $\varphi|_{M^n}$.

The cone $(C(f), \psi)$ of (f, F), given by $C(f)^n = M^{n+1} \oplus A^n$, with

$$d(m, a) = (dm, fm - da) \text{ and } \psi(m, a) = (\varphi m, \varphi a - Fm)$$

is a Tate complex. Moreover, there is a section $\sigma : H^n(C(f)) \to Z^n(C(f))$ in such a way that the diagram

$$\begin{array}{ccc} H^n(C(f)) & \xrightarrow{H(\psi)} & H^n(C(f)) \\ \downarrow \sigma & \overset{\Sigma}{\searrow\!\!\searrow} & \downarrow \sigma \\ Z^n(C(f)) & \xrightarrow{\psi} & Z^n(C(f)) \end{array}$$

commutes up to a homotopy Σ (the proof follows as in the proof of Lemma 4.6). Let $V := H^n(C(f))$ and consider it as a graded vector space of pure degree n. Define a differential $d_V : V \to M^{n+1}$ and a map $f_V : V \to A^n$ by letting $d_V := \pi_1\sigma$ and $f_V := \pi_2\sigma$, where π_i denote the projections from $C(f)^n$ to M^{n+1} and A^n respectively. Note that we have

$$\pi_1\psi = \varphi\pi_1 \text{ and } \pi_2\psi = \varphi\pi_2 - F\pi_1.$$

Using the first of these equations we get

$$\varphi d_V = \varphi\pi_1\sigma = \pi_1\psi\sigma = \pi_1(\sigma H(\psi) - d\Sigma) = \pi_1\sigma H(\psi) = d_V H(\psi)$$

where we used that $\pi_1 d\Sigma = 0$ since Σ lands in the cocycles of $C^n(f)$. Therefore d_V commutes with the endomorphisms. Since $(V, H(\psi))$ is a Tate module, by Lemma 2.9 it has a decomposition $V = \bigoplus V_k$, wherew V_k is the generalized eigenspace of eigenvalue q^k. Furthermore, since d_V is compatible with the endomorphisms, it follows that

$$d_V : V_k \longrightarrow M^{n+1}_k.$$

As a consequence, we obtain a $gr^{(h)}$-weight decomposition on $\tilde{M} := M \sqcup T(V)$, where $\tilde{M}^n_p = M^n_p \oplus V_p$. Lastly, using the above second equation, we get

$$\varphi f_V - f_V H(\psi) = F d_V - \pi_2 d\Sigma$$

and this makes the extension of (f, F) by f_V into a ho-morphism $(\tilde{f}, \tilde{F}) : (\tilde{M}, \varphi) \to (A, \varphi)$. □

Let α be a positive rational number, with $\alpha < h$, where we recall that h denotes the order of q in $\mathbb{F}_\ell^\times$. Our main theorem for cohomological dg-algebras is the following:

Theorem 8.2. *Let $X \in \mathrm{Sch}_K$ be a scheme over K. Assume that for all n, $H^n_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is a pure Tate module of weight αn. Then the following is satisfied:*

(i) *If $\alpha(k-2)/h$ is not an integer then all k-tuple Massey products vanish in $H^*(X_{an}, \mathbb{F}_\ell)$.*

(ii) *If $H^i(X_{an}, \mathbb{F}_\ell) = 0$ for all $0 < i \leq r$ then $H^n(X_{an}, \mathbb{F}_\ell)$ contains no non-trivial Massey products for all $n \leq \lceil \frac{hr}{\alpha} + 2r + 1 \rceil$.*

(iii) *If the graded algebra $H^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is simply connected, then $C^*_{sing}(X_{an}, \mathbb{F}_\ell)$ is an N-formal dg-algebra, with $N = \lfloor (h-1)/\alpha \rfloor$.*

Proof. By assumption, the dg-algebra $C^*_{sing}(X_{an}, \mathbb{F}_\ell)$ is quasi-isomorphic to a monoid in Tate complexes $(C^*_{et}(X, \mathbb{F}_\ell), \varphi)$ with α-pure cohomology. Therefore by Proposition 8.1, it is quasi-isomorphic to a dg-algebra in $gr^{(h)}\mathsf{DGA}^{\alpha\text{-}pure}_{\mathbb{F}_\ell}$. It now suffices to apply Propositions 7.8, 7.4, 7.6 respectively for each of the implications (i), (ii) and (iii). □

We have the following direct application of the previous theorem.

Corollary 8.3. *Let X be a smooth and proper scheme over $\mathcal{O}_K$, the ring of integers of K. Assume that the only eigenvalues of the Frobenius action on $H^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ are powers of q. Then:*

(i) *If $(k-2)/2h$ is not an integer then all k-tuple Massey products vanish in $H^*(X_{an}, \mathbb{F}_\ell)$.*
(ii) *If $H^i(X_{an}, \mathbb{F}_\ell) = 0$ for all $0 < i \leq r$ then $H^n(X_{an}, \mathbb{F}_\ell)$ contains no non-trivial Massey products for all $n \leq 2hr + 2r + 1$.*
(iii) *If the graded algebra $H^*_{et}(X_{\overline{K}}, \mathbb{F}_\ell)$ is simply connected, the dg-algebra $C^*_{sing}(X_{an}, \mathbb{F}_\ell)$ is $2(h-1)$-formal.*

Proof. The conditions of the theorem are satisfied with $\alpha = 1/2$ (see Remark 6.3). □

Example 8.4. We can apply this corollary to $\mathbb{P}^n$. This is a smooth and proper scheme over $\mathbb{Z}$. We can therefore base change it to $\mathbb{Z}_p$ with p a prime number that generates $\mathbb{F}_\ell^\times$ and we deduce that $C^*_{sing}(\mathbb{P}^n_{an}, \mathbb{F}_\ell)$ is $2(\ell-1)$-formal. Note that in fact, complex projective spaces are formal over the integers (see [BB17]).

As illustrated by this example, it often happens that the scheme of interest has a smooth and proper model X over a commutative ring R that is finitely generated over $\mathbb{Z}$. In that case there are infinitely many ways to base change X to a ring of the form $\mathcal{O}_K$ (one for each maximal ideal of R). Those different ways give rise to different values for the parameter h and one should pick the option that yields the largest possible value for h.

Remark 8.5. When working over $\mathbb{Q}_\ell$, we can consider the analogous for Weil modules of Proposition 8.1. Since Weil modules factor through $\mathbb{Z}$-graded objects instead of $(\mathbb{Z}/h\mathbb{Z})$-graded objects, we obtain that if $X \in \mathrm{Sch}_K^{\alpha\text{-}pure}$ and $H^*(X_{an}, \mathbb{Q}_\ell)$ is simply connected, then $C^*_{sing}(X_{an}, \mathbb{Q}_\ell)$ is formal.

Denote by $F_m(\mathbb{A}^d)$ the scheme of configurations of m points in $\mathbb{A}^d$. This is informally described at the point set level as follows

$$F_m(\mathbb{A}^d) = \{(x_1, \ldots, x_m) \in (\mathbb{A}^d)^m | x_i \neq x_j,\ 1 \leq i < j \leq m\}$$

The configuration spaces $F_m(\mathbb{R}^d)$ are known to be formal over $\mathbb{Q}$ for any m and d. However, the question of formality over $\mathbb{F}_\ell$ is open as explained in [Sal18]. Using our machinery we deduce some results of formality over $\mathbb{F}_\ell$ for configuration spaces $F_m(\mathbb{C}^d)$. We will in fact consider the more general problem of the formality of a complement of subspace arrangements.

Definition 8.6. Let V be a d-dimensional K-vector space. We say that a finite set $\{W_i\}_{i\in I}$ subspaces of V is a *good arrangement of codimension c subspaces* if

(i) For each $i \in I$, the subsapce W_i is of codimension c.
(ii) For each $i \in I$, the subspaces $\{W_i \cap W_j\}_{j\neq i}$ of W_i form a good arrangment of codimension c subspaces.

Remark 8.7. In particular the empty set of subspaces is a good arrangement of codimension c subspaces. By induction on the size of I, we see that this condition is well-defined.

Example 8.8. Recall that a set of subspaces of codimension c is said to be in general position if the intersection of n of those subspaces is of codimension $\min(d, cn)$. One easily checks that a set of codimensions c subspaces in general position is a good arrangement. However, the converse does not hold as shown in the following example.

Example 8.9. Take $V = (K^d)^m$ and define, for (i, j) an unordered pair of distinct elements in $\{1, \dots, m\}$, the subspace

$$W_{(i,j)} = \{(x_1, \dots, x_n) \in (K^d)^m, x_i = x_j\}.$$

This collection of codimension d subspaces of V is a good arrangement. However, these subspaces are not in general position if m is at least 3. Indeed, the codimension of $W_{(1,2)} \cap W_{(1,3)} \cap W_{(2,3)}$ is $2d$. The complement $V - \bigcup_{(i,j)} W_{(i,j)}$ is exactly $F_m(\mathbb{A}^d)$.

Lemma 8.10 (c.f. [BE97])**.** *Let V be a d-dimensional vector space over K and $\{W_i\}_{i \in I}$ be a finite collection of subspaces that form a good arrangement of codimension c subspaces. Then $H^n_{et}(V - \bigcup_i W_i, \mathbb{F}_\ell)$ is pure of weight $cn/(2c-1)$. Similarly $H^n_{et}(V - \bigcup_i W_i, \mathbb{Q}_\ell)$ is pure of weight $2cn/(2c-1)$.*

Proof. First we observe that the reason for the factor of 2 difference is explained in Remark 2.6. We do the case of $\mathbb{F}_\ell$. The other case works the same. We proceed by induction on the cardinality of I. When I is empty there is nothing to prove. Assume that the lemma has been proven for $|I| < n$. Let j be an element of I. Let $X = V - \bigcup_{i \neq j} W_i$, let $U = V - \bigcup_i W_i$, then U is an open subscheme of X whose complement is $Z = W_j - \bigcup_{i \neq j} W_i \cap W_j$. Both X and Z are complement of good arrangements of $|I| - 1$ codimension c subspaces, thus the induction hypothesis applies to them. We have a Gysin long exact sequence

$$\dots \to H^{n-2c}_{et}(Z, \mathbb{F}_\ell)(-c) \to H^n_{et}(X, \mathbb{F}_\ell) \to H^n_{et}(U, \mathbb{F}_\ell) \to H^{n+1-2c}_{et}(Z, \mathbb{F}_\ell)(-c) \to \dots$$

By the induction hypothesis, both $H^n_{et}(X, \mathbb{F}_\ell)$ and $H^{n+1-2c}_{et}(Z, \mathbb{F}_\ell)(-c)$ are of weight $cn/(2c-1)$, thus $H^n_{et}(U, \mathbb{F}_\ell)$ is also of weight $2cn/(2c-1)$ as desired. □

Theorem 8.11. *Let X be a complement of a good arrangement of codimension c subspaces defined over K. Then:*

(i) *If $(2c-1)(k-2)/hc$ is not an integer then all k-tuple Massey products vanish in $H^*(X_{an}, \mathbb{F}_\ell)$.*

(ii) *The space $H^n(X_{an}, \mathbb{F}_\ell)$ contains no non-trivial Massey products for*

$$n \leq \left\lceil \frac{h(2c-1)(2c-2)}{c} + 4c - 3 \right\rceil$$

(iii) *If $H^*_{et}(X, \mathbb{F}_\ell)$ is simply connected, then the dg-algebra $C^*_{sing}(X_{an}, \mathbb{F}_\ell)$ is N-formal, with $N = \lfloor (h-1)(2c-1)/c \rfloor$.*

Proof. By Lemma 8.10 we know that codimension c subspace arrangements satisfy the conditions of Theorem 8.2. For part (ii), one shows by induction on the number of subspaces that the cohomology of X vanishes in degree $\leq 2c-2$. □

Remark 8.12. By (i) of the above Theorem we have that for complements of hyperplane arrangements defined over K, all triple Massey products over $\mathbb{F}_\ell$ are trivial, as long as $h > 1$. In [Mat06], Matei showed that for every odd prime ℓ, there is a (non-simply connected) complement of hyperplane arrangements X in $\mathbb{C}^3$ with a non-vanishing triple Massey product in $H^2(X, \mathbb{F}_\ell)$. These two facts do not contradict each other since Matei's hyperplane arrangement cannot be modeled over a p-adic field K with residue field $\mathbb{F}_q$ unless ℓ divides $q-1$ (indeed, it requires K to have all ℓ-th root of unity) but in that case $h = 1$ and Theorem 8.11 is vacuous.

The above result applies to configuration spaces of m points in $\mathbb{C}^d$. We get:

Theorem 8.13. *For any finite field $\mathbb{F}_\ell$ and every $d > 1$, the dg-algebra $C^*_{sing}(F_m(\mathbb{C}^d), \mathbb{F}_\ell)$ is N-formal, with*

$$N = \left\lfloor \frac{(\ell-2)(2d-1)}{d} \right\rfloor.$$

Proof. The space $F_m(\mathbb{C}^d)$ is the complement of a good codimension d subspace arrangement defined over any of the fields $\mathbb{Q}_p$ (in fact it can be defined over $\mathbb{Z}$). We can thus pick a prime number p such that p generates $\mathbb{F}_\ell^\times$ and we get the desired result from the previous theorem. □

Corollary 8.14. *For any finite field $\mathbb{F}_\ell$ and every $d > 1$, the dg-algebra $C^*_{sing}(F_m(\mathbb{C}^d), \mathbb{F}_\ell)$ is formal when*

$$(m-1)d \leq \ell - 2.$$

Proof. The cohomology of $C^*_{sing}(F_m(\mathbb{C}^d), \mathbb{F}_\ell)$ is concentrated in degree $\leq (m-1)(2d-1)$. Therefore, by the previous theorem, this dg-algebra is formal when

$$(m-1)(2d-1) \leq \frac{(\ell-2)(2d-1)}{d}.$$

This is equivalent to asking for $(m-1)d \leq \ell - 2$. □

Example 8.15. Consider the configuration space $F_m(\mathbb{C})$ of m points in $\mathbb{C}$. In [Sal18] it is shown that $C^*_{sing}(F_m(\mathbb{C}), \mathbb{F}_2)$ is not formal for any $m \geq 4$, and the question of whether $C^*_{sing}(F_m(\mathbb{C}), \mathbb{F}_\ell)$ is formal for $\ell > 2$ is left open. Our Theorem 8.13 does not apply in this case, since $F_m(\mathbb{C})$ is not-simply connected. However, (i) of Theorem 8.11 applies to give vanishing results of Massey products: we get, that if $\ell \geq k$, then all k-tuple Massey products vanish in $H^*(F_m(\mathbb{C}), \mathbb{F}_\ell)$. This partially answers a question raised at the end of [Sal18], asking how far one has to go on the filtered model to find obstructions to formality of the dg-algebra $C^*_{sing}(F_m(\mathbb{C}), \mathbb{F}_\ell)$.

Remark 8.16. In forthcoming work by Boavida de Brito and the second author, similar formality results are obtained for the spaces $F_m(\mathbb{R}^d)$ with d not necessarily even. The strategy of proof is the same as the one used here. We build an automorphism on the dg-algebra of singular cochains that plays the role of the Frobenius automorphism. However contrary to the situation here, that automorphism does not come directly from algebraic geometry.

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