THE HOMOTOPY THEORY OF A_∞-CATEGORIES

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Abstract. We put a left model structure on the category of A_∞-categories enriched in a reasonable monoidal model category. The weak equivalences are the Dwyer-Kan equivalences.

Contents

Let $\mathbf{C}$ be a symmetric monoidal combinatorial closed model category satisfying the monoid axiom. Muro proves in [Mur12] that there is a model structure on the category of small $\mathbf{C}$-enriched categories in which the weak equivalences are the Dwyer-Kan equivalences. The purpose of this note is to extend his result to the category of $\mathcal{A}_\infty$-categories. More precisely, we give ourselves a non-symmetric operad $\mathcal{O}$ in $\mathbf{C}$ with a map to $\mathcal{A}$ the associative operad. We assume that the map $\mathcal{O} \to \mathcal{A}$ is a weak equivalence and an epimorphism in arity 0 and 2. Given an $\mathcal{O}$-category C, we can form its homotopy category hC. The definition is given in 2.2.

Definition 0.1. We say that a map $f : C \to D$ of $\mathcal{O}$-categories is a Dwyer-Kan equivalence if

- For each pair (x, y) of objects of C, the induced map $\mathrm{Map}_C(x, y) \to \mathrm{Map}_D(fx, fy)$ is a weak equivalence in $\mathbf{C}$.

The induced map $hC \to hD$ is an equivalence of categories.

Our main results are the following :

Theorem 0.2. *There is a left model structure on $\mathcal{O}\mathbf{Cat}$, the category of $\mathcal{O}$-categories enriched in $\mathbf{C}$. The weak equivalences are the Dwyer-Kan equivalences and the trivial fibrations are the maps that are surjective on objects and induce trivial fibrations on mapping spaces. Moreover, this left model structure is equivalent to the Muro model structure on $\mathcal{A}\mathbf{Cat}$.*

Theorem 0.3. *If $\mathbf{C}$ satisfies the following two extra conditions:*

- *All objects of $\mathbf{C}$ are cofibrant.*
- *The weak equivalences in $\mathbf{C}$ are stable under filtered colimits.*

Then the left model structure on $\mathcal{O}\mathbf{Cat}$ is in fact a model structure that is moreover left proper.

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1. Left model categories

1.1. **Definitions and Smith theorem.** Background on left model categories can be found in [Bar10].

Definition 1.1. A left model category is a category with three subcategories containing all the objects C, F and W satisfying the following axioms:

(1) The category $\mathbf{M}$ is complete and cocomplete.
(2) The class W satisfies the two-out-of three property.
(3) The three classes of maps W, F and C are closed under retracts.
(4) Cofibrations have the left lifting property against trivial fibrations and fibrations have the right lifting property against trivial cofibrations with cofibrant domain.
(5) Any map can be factored as a cofibration followed by a trivial fibration and any map with cofibrant domain can be factored as a trivial cofibration followed by a fibration.

Remark 1.2. As in the case of model categories, an object X is cofibrant if the map $\varnothing \to X$ is a cofibration. In particular, the initial object itself is cofibrant.

Remark 1.3. Let f be a map with the left lifting property against trivial fibrations. Then f can be factored as a cofibrations followed by a trivial fibrations. According to the retract argument, f is a retract of the first map in the factorization hence, f is a cofibration. Similarly, if f has cofibrant domain and the left lifting property against fibrations, then f is a trivial cofibration. Dually, one proves that if f has cofibrant domain and has the right lifting property against trivial cofibration with cofibrant domain, then f is a fibration and if f has the right lifting property against cofibrations, then f is a trivial fibration.

Let $\mathbf{M}$ be a complete and cocomplete category and let I be a set of maps of $\mathbf{M}$.

Definition 1.4.

- An I-injective map is a map with the right lifting property against maps of I.
- An I-cofibration is a map with the left lifting property against I-injective maps.
- An I-cell complex is a transfinite composition of pushouts of maps of I.
- An I-cofibrant object is an object X such that $\varnothing \to X$ is an I-cofibration.

We denote by I-inj the class of I-injective maps. We denote by I-cof the class of I-cofibrations.

Remark 1.5. The class I-cof can be alternatively defined as the smallest weakly saturated class of arrows of $\mathbf{M}$ containing I. It is also the class of arrows which can be obtained as a retract of an I-cell complex. These results can be fond in appendix A of [Lur09].

We have the following theorem which allows us to construct left model structures. The analogous result for model structure is well-known.

Theorem 1.6. *Let $\mathbf{M}$ be a complete and cocomplete category with a class W of maps which contains all the isomorphisms and is closed under retracts and under the two-out-of three property. Let I and J be two sets of maps. Assume that*

- *Both I and J permit the small object argument.*
- *I-inj is a subclass of J-inj $\cap W$.*
- *Elements of J are in I-cof.*
- *J-cell complexes with cofibrant domain are weak equivalences.*
- *Either*
  - *J-inj $\cap W$ is a subclass of I-inj.*
  - *I-cof $\cap W$ is a subclass of J-cof.*

Then there is a left model structure on $\mathbf{M}$ *in which the weak equivalences are the maps of* W*, the cofibrations are the* I*-cofibrations and the fibrations are the* J*-injective.*

Proof. Axiom 1, 2 are satisfied by hypothesis. W is closed under retracts by assumption. F and C are closed under retracts since they are defined by a right or left lifting property.

The small object argument gives a factorization of any map as a cofibration followed by a trivial fibration and as a relative J-cell complex followed by a fibration.

By assumption, a relative J-cell complex is a cofibration. Since by assumption, a J-cell complex with cofibrant domain is a weak equivalence, axiom 5 is satisfied.

Now let us check axiom 4 assuming the second case of the last hypothesis. By definition a cofibration has the left lifting property against any map in I-inj. Since I-inj is exactly the class of trivial fibration the first half of the axiom is checked.

Let $f : A \to B$ be a trivial cofibration with cofibrant domain, we want to show that f is in J-cof. By the small object argument, we can factor $f = p \circ i$ as a relative J-cell complex i followed by a map in J-inj. A relative J-cell complex with cofibrant domain is a weak equivalence, thus p is a weak equivalence. According to our hypothesis, p is in I-inj. According to the retract argument, f is a retract of i. Hence f is in J-cof. □

If the category is locally presentable, there is a simplification of this theorem due to Jeff Smith in the case of model structures. We give a version in the case of left model structures.

Theorem 1.7. *Let* $\mathbf{M}$ *be a locally presentable category,* W *be a subset of* $\mathrm{Ar}(\mathbf{M})$ *containing all identity maps. Let* I *be a set of morphisms of* $\mathbf{M}$ *such that:*

- W *is an accessible subcategory.*
- I*-inj is contained in* W*.*
- *An* I*-cof* $\cap W$*-cell complex with cofibrant codomain is in* W*.*

Then $\mathbf{M}$ *has a left model category structure whose weak equivalences are maps in* W*, and cofibrations are maps in* I*-cof.*

Proof. We mimick the proof of the analgous result in [Bar10]. [Bar10, Lemma 2.4.] holds without change with our hypothesis and gives a set J of arrows in $W \cap I$-cof. [Bar10, Lemma 2.3.] tells us that $W \cap I$-cof is a subclass of J-cof. Let us check that this set J satisfies the hypothesis of 1.6. By hypothesis, the pushout of a map in J along a map with cofibrant codomain is in W. Let f be a map in I-inj, we already know that f is in W, we need to check that it is in J-inj but this is clear since J is contained in I-cof. □

2. A-infinity categories

Let $\mathbf{C}$ be a symmetric monoidal category and $\mathcal{O}$ be a non-symmetric operad in $\mathbf{C}$.

Definition 2.1. An $\mathcal{O}$-category C is the data of:

- A set of object $\mathrm{Ob}(C)$.
- For each pair of objects (x, y) an object $\mathrm{Map}(x, y)$ of $\mathbf{C}$.
- Composition maps:

$$\mathcal{O}(n) \otimes \mathrm{Map}(x_0, x_1) \otimes \mathrm{Map}(x_1, x_2) \otimes \ldots \otimes \mathrm{Map}(x_{n-1}, x_n) \to \mathrm{Map}(x_0, x_n)$$

moreover, these compositions maps are assumed to be associative in the obvious way.

A map of $\mathcal{O}$-categories $C \to D$ is the data of a map $f : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ and a map $\mathrm{Map}_C(x, y) \to \mathrm{Map}_D(fx, fy)$ for all pair of objects (x, y) in $\mathrm{Ob}(C)$ that is compatible with the composition in the obvious way.

Let $\mathcal{A}$ be the associative operad. It is the non-symmetric operad with $\mathcal{A}(n) = \mathbb{I}$. An algebra over $\mathcal{A}$ is nothing but a monoid in $\mathbf{C}$. In that case, an $\mathcal{A}$-category is just a category enriched in $\mathbf{C}$.

The definition of an $\mathcal{O}$-category can be made for an arbitrary non-symmetric operad $\mathcal{O}$. However, from now on we will assume that $\mathbf{C}$ has the structure of a symmetric monoidal model category and that $\mathcal{O}$ is equipped with a map $\mathcal{O} \to \mathcal{A}$ which is a weak equivalence of operad. This assumption will be necessary to be able to speak about the homotopy category of an $\mathcal{O}$-category in a sensible way.

Definition 2.2. Let C be an $\mathcal{O}$-category. The homotopy category hC is the category whose set of object is $\mathrm{Ob}(C)$ and with

$$hC(x, y) = \mathrm{Ho}(\mathbf{C})(\mathbb{I}, \mathrm{Map}_C(x, y))$$

Example 2.3. Let X be a topological space. Let $\mathcal{E}_1$ be the operad of little intervals. There is an $\mathcal{E}_1$-category $\mathbf{P}(X)$ whose objects are the points of X and whose space of morphisms between two points (x, y) is the pullback

$$\begin{tikzcd}
\mathbf{P}(X)(x,y) \arrow[r] \arrow[d] & \mathrm{Map}([0,1], X) \arrow[d] \\
\mathrm{pt} \arrow[r, "{(x,y)}"] & \mathrm{Map}(\{0,1\}, X) \cong X \times X
\end{tikzcd}$$

Let S be a set. We denote by $\mathcal{O}\mathbf{Cat}_S$ the category whose objects are $\mathcal{O}$-categories whose set of objects is S and whose morphisms are the morphisms of $\mathcal{O}$-categories that induce the identity on the set of objects.

If $\mathbf{C}$ is a monoidal model category satisfying the monoid axiom, the transferred model structure exists on $\mathcal{O}\mathbf{Cat}_S$ for each S (see [Mur11, Corollary 10.4 and 10.5]). Moreover any map of sets $S \to T$ induces a map of operad $\mathcal{O}_S \to \mathcal{O}_T$ which correspondingly induces a Quillen adjuction

$$u_! : \mathcal{O}\mathbf{Cat}_S \leftrightarrows \mathcal{O}\mathbf{Cat}_T : u^*$$

We will need to make an extra-assumption on the map $\mathcal{O} \to \mathcal{A}$. First, recall that a map $c \to d$ in a category $\mathbf{C}$ is said to be an epimorphism if for all k in $\mathbf{C}$ the induced map

$$\mathbf{C}(d, k) \to \mathbf{C}(c, k)$$

is an injection.

Proposition 2.4. *Let $\mathbf{C}$ be a closed symmetric monoidal category. Let $p : c \to d$ be an epimorphism and k be any object then $p \otimes \mathrm{id}_k$ is an epimorphism.*

Proof. Let l be an object of $\mathbf{C}$, we need to check that

$$\mathbf{C}(d \otimes k, l) \to \mathbf{C}(c \otimes k, l)$$

is an injection.

Let Hom denotes the inner Hom of $\mathbf{C}$. By adjunction, the above map can be identified with

$$\mathbf{C}(d, \mathrm{Hom}(l, k)) \to \mathbf{C}(c, \mathrm{Hom}(l, k))$$

which is an injection since p is an epimorphism. □

From now, on, we will make the following assumption about the map $\mathcal{O} \to \mathcal{A}$.

Assumption 2.5. The map $\mathcal{A}(2) \to \mathcal{O}(2)$ and $\mathcal{O}(0) \to \mathcal{A}(0)$ are epimorphisms.

3. The left model structure

3.1. **The cofibrations.** This subsection is inspired by the paper [Sta12] which explains how to construct weak factorization systems on a Grothendieck construction.

We make the following notations. Let $F : \mathbf{K} \to \mathbf{Cat}$ be a functor with the property that $F(u)$ is a left adjoint for any map $u \in \mathbf{K}$. We denote by $u_!$ the functor $F(u) : F(k) \to F(l)$ and by u^* its right adjoint.

The objects in the Grothendieck construction $Gr(F)$ are denoted (k, X) where k is an object of $\mathbf{K}$ and X is an object in $F(k)$. A map in the Grothendieck construction of F from (k, X) to (l, Y) can be described as a pair (u, f^u) where $u : k \to l$ is a map in $\mathbf{K}$ and $f^u : X \to u^*Y$ or equivalently as a pair (u, f_u) where $f_u : u_!X \to Y$ is the adjoint map to f^u.

Proposition 3.1. *Assume that* $\mathbf{K}$ *is equipped with a weak factorization system* $(\mathcal{A}, \mathcal{B})$*. Assume further that for each* $k \in \mathbf{K}$*,* $F(k)$ *is endowed with a weak factorization system* $(\mathcal{A}_k, \mathcal{B}_k)$*. Assume that for any map* $u : k \to l$*, the functor* u^* *sends* $\mathcal{B}_l$ *to* $\mathcal{B}_k$*. Then there is a weak factorization system* $(\mathcal{A}^p, \mathcal{B}^p)$ *on* $Gr(F)$ *in which a map* (u, f_u) *is in* $\mathcal{A}_p$ *if* u *is in* $\mathcal{A}$ *and* f_u *is in* $\mathcal{A}_l$ *and a map* (u, f^u) *is in* $\mathcal{B}^p$ *if* u *is in* $\mathcal{B}$ *and* f^u *is in* $\mathcal{B}_k$*.*

Proof. See [Sta12]. □

The category of $\mathcal{O}$-categories can be described as the Grothendieck construction of the functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ sending S to the category $\mathcal{O}\mathbf{Cat}_S$. We denote by $(u, f^u) : C \to D$ a map of $\mathcal{O}$-category where $u : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ is a map of sets and $f^u : C \to u^*D$ is a map of $\mathcal{O}$-categories with $\mathrm{Ob}(C)$ as set of objects.

Definition 3.2. We say that a map (u, f^u) of $\mathcal{O}$-categories $C \to D$ is a trivial fibration if u is surjective and f^u is a trivial fibration. We say that a map is a cofibration if u is injective and f_u is a cofibration.

Proposition 3.3. *Cofibrations and trivial fibrations form a weak factorization system on* $\mathcal{O}\mathbf{Cat}$*.*

Proof. This follows from 3.1. □

Observe that a cofibrant object in $\mathcal{O}\mathbf{Cat}$ or $\mathcal{A}\mathbf{Cat}$ is just an object which is cofibrant in $\mathcal{O}\mathbf{Cat}_S$ or $\mathcal{A}\mathbf{Cat}_S$ where S is its set of objects.

Proposition 3.4. *A trivial fibration* $C \to D$ *is a map such that* $\mathrm{Ob}(C) \to \mathrm{Ob}(D)$ *is surjective and for each* x, y *in* $\mathrm{Ob}(C)$*, the induced map*

$$\mathrm{Map}_C(x, y) \to \mathrm{Map}_D(x, y)$$

is a trivial fibration in $\mathbf{C}$*.*

Proof. Trivial. □

For K an object of $\mathbf{C}$, we denote by $[1]_K$ the object of $\mathcal{O}\mathbf{Cat}_{0,1}$ with

$$\mathrm{Map}_{[1]_K}(0, 1) = K,\ \mathrm{Map}_{[1]_K}(1, 0) = \varnothing,\ \mathrm{Map}_{[1]_K}(0, 0) = \mathrm{Map}_{[1]_K}(1, 1) = \mathbb{I}$$

Note that for any C in $\mathcal{O}\mathbf{Cat}$, a map $[1]_K \to C$ is exactly the data of two points (x, y) in $\mathrm{Ob}(C)$ and a map $K \to \mathrm{Map}_C(x, y)$.

We define a set I of arrows in $\mathcal{O}\mathbf{Cat}$ consisting of the inclusion $\varnothing \to [0]$ and the maps $[1]_K \to [1]_L$ for $\{K \to L\}$ a set of generating cofibrations in $\mathbf{C}$. using the previous proposition, it is clear that a map with the RLP against those maps is exactly a trivial fibration.

There is a left adjoint functor $S : \mathcal{O}\mathbf{Cat}_T \to \mathcal{A}\mathbf{Cat}_T$ for each T which is a left Quillen equivalence if $\mathcal{O}$ is equivalent to the operad $\mathcal{A}$. These left adjoint assemble into a left adjoint $S : \mathcal{O}\mathbf{Cat} \to \mathcal{A}\mathbf{Cat}$ which we call strictification. We denote by U its right adjoint. U can be described simply as the functor which gives to a category the trivial $\mathcal{O}$-category structure transferred along the morphism $\mathcal{O} \to \mathcal{A}$. Note that U and S preserve the set of objects.

Proposition 3.5. *The functor S preserves cofibrations.*

Proof. Let $u : S \to T$ be a map of sets. The diagram of right adjoints

$$\begin{tikzcd} \mathcal{A}\mathbf{Cat}_T \arrow[r, "u^*"] \arrow[d, "U"] & \mathcal{A}\mathbf{Cat}_S \arrow[d, "U"] \\ \mathcal{O}\mathbf{Cat}_T \arrow[r, "u^*"] & \mathcal{O}\mathbf{Cat}_S \end{tikzcd}$$

obviously commutes up to isomorphism. Therefore, the corresponding diagram of left adjoints

$$\begin{tikzcd} \mathcal{O}\mathbf{Cat}_S \arrow[r, "u_!"] \arrow[d, "S"] & \mathcal{O}\mathbf{Cat}_T \arrow[d, "S"] \\ \mathcal{A}\mathbf{Cat}_S \arrow[r, "u_!"] & \mathcal{A}\mathbf{Cat}_T \end{tikzcd}$$

commutes up to isomorphism. Let $f = (u, f_u)$ be a cofibration in $\mathcal{O}\mathbf{Cat}$. Strictification preserves the set of object, therefore, $S(f)$ is injective on objects. According to the previous remark, the proposition will be proved if we show that the strictification of a cofibration in $\mathcal{O}\mathbf{Cat}_T$ is a cofibration in $\mathcal{A}\mathbf{Cat}_T$. But this follows from the fact that $S : \mathcal{O}\mathbf{Cat}_T \to \mathcal{A}\mathbf{Cat}_T$ is a left Quillen functor. □

3.2. **Weak equivalences.**

Definition 3.6. We say that a map between cofibrant objects of $\mathcal{O}\mathbf{Cat}$ is a weak equivalence if its strictification is a weak equivalence. We say that a map $C \to D$ is a weak equivalence if it is so when one applies a cofibrant replacement functor to it.

Our goal is to show that the class of weak equivalences coincides with the class of DK equivalences defined in 0.1. Notice first that both classes of equivalences satisfy the two-out-of three condition.

Lemma 3.7. *Let $C \to D$ be a map in $\mathcal{O}\mathbf{Cat}_S$. Then $C \to D$ is a weak equivalence if and only if it is a DK equivalence.*

Proof. With a fixed set of object, the essential surjectivity is automatic. The result is then obvious. □

Lemma 3.8. *$C \to D$ is a DK equivalence of $\mathcal{A}$-categories if and only if $U(C) \to U(D)$ is a DK equivalence of $\mathcal{O}$-categories.*

Proof. Indeed, the homotopy category of $U(C)$ is isomorphic to the homotopy category of C. □

Proposition 3.9. *The functor U is fully faithful.*

Proof. It is clear that U is faithful. If $f : UC \to UD$ is a map of $\mathcal{O}$-categories, then it induces a map

$$\mathrm{Map}_C(x, y) \to \mathrm{Map}_D(fx, fy)$$

for any pair of objects x and y. We claim that these maps assemble into a map in $\mathcal{A}\mathbf{Cat}$. Indeed, if x, y, z are three objects of C, we have a diagram

$$\begin{tikzcd}
\mathcal{O}(2)\otimes \mathrm{Map}_C(x,y)\otimes \mathrm{Map}_C(y,z) \arrow[r, "f"] \arrow[d] & \mathcal{O}(2)\otimes \mathrm{Map}_D(fx,fy)\otimes \mathrm{Map}_D(fy,fz) \arrow[d] \\
\mathrm{Map}_C(x,y)\otimes \mathrm{Map}_C(y,z) \arrow[r, "f"] \arrow[d] & \mathrm{Map}_D(fx,fy)\otimes \mathrm{Map}_D(fy,fz) \arrow[d] \\
\mathrm{Map}_C(x,y) \arrow[r, "f"] & \mathrm{Map}_D(fx,fy)
\end{tikzcd}$$

The top square obviously commutes. The total square commutes because $f : UC \to UD$ is a map of $\mathcal{O}$-categories. Since the map

$$\mathcal{O}(2)\otimes \mathrm{Map}_C(x,y)\otimes \mathrm{Map}_C(y,z) \to \mathrm{Map}_C(x,y)\otimes \mathrm{Map}_C(y,z)$$

is an epimorphism by our assumption 2.5 and 2.4, we find that the bottom square commutes, that is f commutes with the composition of C and D. One would show similarly that f commutes with the units of C and D using the fact that $\mathcal{O}(0) \to \mathcal{A}(0)$ is an epimorphism. Hence, f lifts to a map in $\mathcal{A}\mathbf{Cat}$. □

Corollary 3.10. *Let C be an $\mathcal{A}$-category. The counit map $SU(C) \to C$ is an isomorphism.*

Proposition 3.11. *A map in $\mathcal{O}\mathbf{Cat}$ is a weak equivalence if and only if it is a DK equivalence.*

Proof. If C is a cofibrant $\mathcal{O}$-category, then the unit map $C \to US(C)$ is a weak equivalence in $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(C)}$ hence, it is a DK equivalence by 3.7.

Let $C \to D$ be a map of $\mathcal{O}$-category with C and D cofibrant. We have a commutative diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[d] \\
US(C) \arrow[r] & US(C)
\end{tikzcd}$$

The vertical maps are DK equivalences and weak equivalences. Thus the upper horizontal map is a DK equivalence (resp. weak equivalence) if and only if the lower one is. The result then follows from 3.8.

Now let $C \to D$ be a map between not necessarily cofibrant $\mathcal{O}$-categories. Let $C' \to C$ be a cofibrant replacement of C in $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(C)}$. The composite map $g : C' \to C \to D$ can be factored as

$$C' \to u_!C' \to D$$

where $u : \mathrm{Ob}(C) = \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ is the induced map on objects. Since $u_!$ is left Quillen, then $u_!C'$ is cofibrant in $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(D)}$. We factor this map further into

$$C' \to u_!C' \to D' \to D$$

where $u_!C' \to D'$ is a cofibration in $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(D)}$ and $D' \to D$ is a trivial fibration. In particular, D' is a cofibrant object fitting in the commutative diagram

$$\begin{tikzcd}
C' \arrow[r] \arrow[d] & D' \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in which both vertical arrows are weak equivalences in $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(C)}$ and $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(D)}$. This implies by 3.7 that both vertical arrows are DK equivalences and weak equivalences in $\mathcal{O}\mathbf{Cat}$. Hence the lower horizontal arrow is a weak equivalence (resp. DK equivalence) if and only if the upper horizontal arrow is a weak equivalence (resp. DK equivalence). This concludes the proof. □

3.3. **Existence of the left model structure.**

Proposition 3.12. *The maps in I-inj are weak equivalences.*

Proof. By 3.11, it suffices to show that the maps in I-inj are DK-equivalences.

We have already mentioned that a map in I-inj is a map $f = (u, f^u) : C \to D$ such that u is surjective and f^u is a trivial fibration in $\mathcal{O}\mathbf{Cat}_T$ where T is the domain of u. In particular, f is essentially surjective. Let x and y be two elements of T, let $i : \{x, y\} \to T$ be the inclusion. The map

$$i^* f^u : i^* C \to i^* u^* D$$

is a trivial fibration in $\mathcal{O}\mathbf{Cat}_{\{x,y\}}$ since i^* is a right Quillen adjoint. This means in particular that the map

$$\mathrm{Map}_C(x, y) \to \mathrm{Map}_D(ux, uy)$$

is a trivial fibration. This proves that f is a DK equivalence. □

Proof of 0.2. By 1.7, it suffices to prove that an I-*cof* $\cap\, W$-cell complex with cofibrant domain is a weak equivalence. Let $f : A \to B$ be such a map. Since f is a cofibration, then B is also cofibrant. To check that f is a weak equivalence it suffices to show that its strictification is a weak equivalence. The map f is a transfinite composition of pushouts of maps of I. Since the domain of f is cofibrant, and pushout and transfinite compostion of maps of I are cofibrations, each step in the transfinite composition is cofibrant. The functor S preserves cofibration and weak equivalences between cofibrant objects. As a left adjoint, S also preserves pushouts and transfinite compositions. Thus $S(f)$ is a transfinite composition of pushouts of maps which are weak equivalences and cofibrations in $\mathcal{A}\mathbf{Cat}$. Since $\mathcal{A}\mathbf{Cat}$ is a model category by [Mur12, Theorem 1.1.], $S(f)$ is a trivial cofibration. □

3.4. **Model structure.** In this subsection, we assume that all objects of $\mathbf{C}$ are cofibrant. Our goal is to prove that the left model structure of the previous proposition is in fact a model structure.

Lemma 3.13. *Let $K \to L$ be a generating cofibration in $\mathbf{C}$. Let us consider a commutative diagram*

$$\begin{tikzcd} {[1]_K} \arrow[r, "g"] \arrow[d] & C \arrow[r, "u"] \arrow[d] & D \arrow[d] \\ {[1]_L} \arrow[r] & C' \arrow[r, "u"'] & D' \end{tikzcd}$$

in which both squares are pushout squares. Then if u is a weak equivalence which is a bijection on objects, so is u'.

Proof. We denote by $f : \{0, 1\} \to S = \mathrm{Ob}(C)$ the map induced by g on objects. We have the object $f_![1]_K$ in $\mathcal{O}\mathbf{Cat}_S$ which is cofibrant. Moreover, the map $f_![1]_K \to f_![1]_L$ is a cofibration in $\mathcal{O}\mathbf{Cat}_S$. We also have a commutative diagram

$$\begin{tikzcd} f_![1]_K \arrow[r] \arrow[d] & C \arrow[r, "u"] \arrow[d] & D \arrow[d] \\ f_![1]_L \arrow[r] & C' \arrow[r, "u'"'] & D' \end{tikzcd}$$

in which both squares are pushouts. But the map u' is a weak equivalence in $\mathcal{O}\mathbf{Cat}_S$ by [Mur14, Theorem 1.13]. ☐

Now, we can prove theorem 0.3.

Proof of 0.3. We apply [Lur09, Proposition A.2.6.13]. Condition (1) is proved as in the case of $\mathcal{A}\mathbf{Cat}$. Condition (3) is 3.12. Thus it suffices to prove condion (2). Let us consider the following commutative diagram

$$\begin{tikzcd} {[1]_K} \arrow[r] \arrow[d] & C \arrow[r, "u"] \arrow[d] & D \arrow[d] \\ {[1]_L} \arrow[r] & E \arrow[r, "v"'] & F \end{tikzcd} \tag{3.1}$$

in which both squares are pushouts squares and u is a weak equivalence. We want to prove that v is a weak equivalence.

Let $C' \to C$ be a cofibrant replacement of C and $C' \xrightarrow{u'} D' \to D$ be a factorization of $C' \to C \to D$ as a cofibration followed by a trivial fibration. We can assume that $C' \to C$ and $D' \to D$ are cofibrant replacement in $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(C)}$ and $\mathcal{O}\mathbf{Cat}_{\mathrm{Ob}(D)}$respectively. Since K is cofibrant (as is any object in $\mathbf{C}$), we can lift $[1]_K \to C$ to a map $[1]_K \to C'$. We can form the following commutative diagram:

$$\begin{tikzcd} {[1]_K} \arrow[r] \arrow[d] & C' \arrow[r, "u'"] \arrow[d] & D' \arrow[d] \\ {[1]_L} \arrow[r] & E' \arrow[r, "v'"'] & F' \end{tikzcd} \tag{3.2}$$

in which both squares are pushouts. This commutative diagram maps to the previous one. Moreover, according to the previous lemma, the map from diagram 3.2 to diagram 3.1 is levelwise a weak equivalence. Hence, v is a weak equivalence, if and only if v' is one.

We can hit the diagram 3.2 with the functor S and we get a commutative diagram in $\mathcal{A}\mathbf{Cat}$:

$$\begin{tikzcd} {[1]_K} \arrow[r] \arrow[d] & SC' \arrow[r, "Su'"] \arrow[d] & SD' \arrow[d] \\ {[1]_L} \arrow[r] & SE' \arrow[r, "Sv'"'] & SF' \end{tikzcd}$$

in which both squares are pushout squares. Since C' and D' are cofibrant, then Su' is a weak equivalence by definition. Thus, by left properness of $\mathcal{A}\mathbf{Cat}$, we find that Sv' is a weak equivalence. Since E' and F' are cofibrant, this implies that v' is a weak equivalence. ☐

3.5. **Equivalence with strict categories.** The functor $U : \mathcal{A}\mathbf{Cat} \to \mathcal{O}\mathbf{Cat}$ preserves weak equivalences. Thus, it is a map of relative categories.

Theorem 3.14. *The functor $U : \mathcal{A}\mathbf{Cat} \to \mathcal{O}\mathbf{Cat}$ is a weak equivalence of relative categories.*

Proof. Let us consider the functor $S \circ Q : \mathcal{O}\mathbf{Cat} \to \mathcal{A}\mathbf{Cat}$ where Q is a functorial cofibrant replacement functor in $\mathcal{A}\mathbf{Cat}$. Then $S \circ Q$ preserves weak equivalences. We also have natural zig-zags of weak equivalences $S \circ Q \circ U \to S \circ U = \mathrm{id}_{\mathcal{O}\mathbf{Cat}}$ and $\mathrm{id}_{\mathcal{A}\mathbf{Cat}} \leftarrow Q \to U \circ S \circ Q$. The reason for these two zig-zag of weak equivalences is that they exist in $\mathcal{A}\mathbf{Cat}_S$ and $\mathcal{O}\mathbf{Cat}_S$ for each S because

$$S : \mathcal{O}\mathbf{Cat}_S \leftrightarrows \mathcal{A}\mathbf{Cat}_S : U$$

is a Quillen equivalence in which U preserves all weak equivalences. ☐

4. Examples

Bicategories. Take **Cat** to be the category of categories and $\mathcal{O}$ to be an appropriate operad in **Cat**. Then $\mathcal{O}$**Cat** is the category of bicategories. We can put the canonical model structure on **Cat**. The weak equivalences are the equivalences of categories. Wih this model structure, **Cat** satisfies the conditions of 0.3, thus, we find a model structure on the category of bicategories which is Quillen equivalent to the model structure on 2-categories. This recovers the main result of [Lac04].

A_∞**-categories.** Take $\mathbf{Ch}(R)$ to be the category of chain complexes over a commutative ring with the projective model structure. Take $\mathcal{O}$ to be a non symmetric operad in $\mathbf{Ch}(R)$ equipped with a weak equivalence to the associative operad. Then there is a left model structure on the category of $\mathcal{O}$-categories. In particular, taking $\mathcal{O}$ to be the cellular chains on the Stasheff operad, we obtain the well-studied notion of an A_∞-category. Theorem 0.3 insures that if R is a field, this left model structure is in fact a model structure.

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