

EQUIVARIANT FORMALITY OF THE LITTLE DISKS OPERAD

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ABSTRACT. The little n -disks operad is $SO(n)$ and $O(n)$ -equivariantly formal over the rationals. Equivalently, the oriented and unoriented framed little disks operads are rationally formal as ∞ -operads.

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The little disks operad \mathcal{D}_n of dimension n has a natural action of the orthogonal group $O(n)$, by rotating disks. In other words, \mathcal{D}_n is an operad in $O(n)$ -spaces. In this paper, we show that it is formal as such. To explain what this means precisely, recall that the equivariant homotopy type of a space X equipped with an action of a group G can be recorded in the Borel construction X_{hG} together with its canonical map to $BG = *_hG$. The Borel construction defines an equivalence of ∞ -categories from G -spaces (up to weak equivalence) to spaces over BG . In particular, applying the Borel construction aritywise to an operad \mathcal{P} in G -spaces produces an operad in the category of spaces over BG . The product in the latter category is the fiber product over BG . Very concretely, the operadic structure is given by composition maps (over BG)

$$\circ_i : \mathcal{P}(n)_{hG} \times_{BG} \mathcal{P}(m)_{hG} \rightarrow \mathcal{P}(n + m - 1)_{hG}$$

satisfying the usual axioms.

For a finite type nilpotent space X , the rational localization of X can be recovered from $\mathcal{A}_{\text{pl}}(X)$, where \mathcal{A}_{pl} denotes Sullivan's functor of piece-wise linear forms from spaces to commutative dg-algebras. When trying to define formality of operads from this perspective, one runs into the problem that \mathcal{A}_{pl} is not symmetric monoidal on the nose. Instead, there is a natural map

$$\mathcal{A}_{\text{pl}}(X) \otimes \mathcal{A}_{\text{pl}}(Y) \rightarrow \mathcal{A}_{\text{pl}}(X \times Y)$$

which is a quasi-isomorphism. This implies that, if \mathcal{P} is a topological operad, $\mathcal{A}_{\text{pl}}(\mathcal{P})$ is not an operad (in the opposite of the category of commutative dg-algebras) on the nose but only *up to coherent homotopies*. There are known ways to resolve or circumvent this. A convenient way is using

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the theory of ∞ -operads through the dendroidal formalism [CM13]. Given a topological operad \mathcal{P} , one may form its dendroidal nerve which is a functor $N_d\mathcal{P} : \text{Tree}^{\text{op}} \rightarrow \mathbf{S}$ that captures all the homotopical information of the operad \mathcal{P} . Here Tree is the dendroidal category, also denoted Ω in the literature. We then say that an operad \mathcal{P} is *rationaly formal* if $\mathcal{A}_{\text{pl}}(N_d\mathcal{P})$ is formal as a diagram from Tree to commutative dg-algebras. This notion translates well to the equivariant setting. If our operad \mathcal{P} comes with an action of a topological group G , we say that it is *equivariantly formal* if the diagram $\mathcal{A}_{\text{pl}}(N_d(\mathcal{P}_{hG}))$ from Tree to commutative dg-algebras is formal. Here \mathcal{P}_{hG} is the operad in the category of spaces over BG described above. In fact, its dendroidal nerve $N_d(\mathcal{P}_{hG})$ agrees with $(N_d\mathcal{P})_{hG}$, the homotopy orbits of the ∞ -operad associated to \mathcal{P} . Having explained the terminology, we now state the main theorem.

Theorem. *Let $n \geq 3$, let G be $O(n)$ or $SO(n)$. The little n -disks operad \mathcal{D}_n is G -equivariantly formal.*

As a consequence, a rational model for the Borel construction of configuration spaces in \mathbb{R}^n with their $O(n)$ or $SO(n)$ actions is given by their cohomology rings. We compute these in Section 7.

The homotopy orbits $(\mathcal{D}_n)_{hG}$ is an ∞ -operad (a complete dendroidal Segal space, i.e. a fibrant object in a model category describing the homotopy theory of ∞ -operads). This ∞ -operad is in fact a fibrant replacement of the framed little disks (Proposition 2.2). So our main theorem can be rephrased as follows.

Theorem. *In dimension $n \geq 3$, the framed little disks operad and its unoriented variant are formal as ∞ -operads.*

This contrasts with the results of Khoroshkin and Willwacher [KW17] and Moriya [Mor19] who prove that the framed little disks operad is not formal in odd dimensions. This is not a contradiction, but can be explained by the fact that the dendroidal nerve of the framed little disks operad is not an ∞ -operad: it satisfies the Segal condition but it is not complete (see also Remark 2.3).

We also prove a relative version of the above:

Theorem. *Let $n \geq 3$ and $c \geq 2$. Let $G = O(n)$ or $G = SO(n)$. The map $\mathcal{D}_n \rightarrow \mathcal{D}_{n+c}$ induced by the standard inclusion of \mathbb{R}^n in \mathbb{R}^{n+c} is G -equivariantly formal.*

One motivation for such a theorem comes from embedding calculus. Embedding calculus tries to approximate spaces of embeddings between smooth manifolds by studying their effect on disks, i.e. replacing each manifold by the associated right module over the *framed* little disks ∞ -operad. Under the assumption that the manifolds are *parallelized*, the framed little disks ∞ -operad can be replaced by the standard little disks ∞ -operad and, in this passage, the (by now classical) formality has been used to great effect, e.g. [ALV07]. The formality results that we prove in this paper should allow us to dispense such parallelizability assumptions and draw non-trivial consequences for the rational homotopy theory of more general embedding spaces.

Our strategy to prove these results is to use a “purity implies formality” technique. As input, we consider the action of the Grothendieck-Teichmüller group on the rationalized little disks operad, built from the additivity theorem. This action is compatible with the action of the maximal torus of $O(n)$ and so induces an action on the rationalized Borel construction. As a byproduct of our strategy we also obtain rational formality for algebraic group actions on smooth complex projective varieties, using as input the Galois action on étale cohomology.

Related work. A different description of the $SO(n)$ -action on the real homotopy type of \mathcal{D}_n is given in [KW17]. The translation between our result and the main result of [KW17] does not seem to be straightforward as our ways of modelling equivariant homotopy type of operads are quite different. The computation of Section 7, at least with real coefficients, can also be deduced from [KW17] as was explained to us by Thomas Willwacher.

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Convention. All cohomology is with coefficients in \mathbb{Q} , unless stated otherwise.

1. EQUIVARIANT HOMOTOPY THEORY

1.1. Borel construction and rational equivariant homotopy theory. For a topological group G acting on a space X , one may form the Borel construction X_{hG} . This is a space with a map to BG . Explicitly, this can be constructed as

$$X_{hG} := (X \times EG)/G \rightarrow (* \times EG)/G := BG$$

where EG denotes any contractible CW complex with free G -action and the quotient is taken with respect to the diagonal G -action. It is classical that the homotopy type of X with its G action can be reconstructed from this map. Indeed, we have a fiber sequence

$$X \rightarrow X_{hG} \rightarrow BG$$

and the monodromy action of $G \simeq \Omega BG$ coincides with the G -action on X . This can be expressed by the following theorem.

Theorem 1.1. *The Borel construction defines an equivalence of ∞ -categories*

$$\mathrm{Fun}(BG, \mathcal{S}) \rightarrow \mathcal{S}/BG,$$

where \mathcal{S} denotes the ∞ -category of spaces.

Given a G -space X , it is customary to denote by $H_G^*(X)$ the cohomology of the Borel construction. This is called the *equivariant cohomology* of X . Likewise, the *rational equivariant homotopy type* is defined to be the homotopy type of the map of commutative dg-algebras

$$\mathcal{A}_{\mathrm{pl}}(BG) \rightarrow \mathcal{A}_{\mathrm{pl}}(X_{hG}).$$

Definition 1.2. We say that a G -space X is *equivariantly formal* if the map

$$\mathcal{A}_{\mathrm{pl}}(BG) \rightarrow \mathcal{A}_{\mathrm{pl}}(X_{hG})$$

is formal.

Remark 1.3. The above notion of equivariant formality implies the multiplicative collapse of the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{H^*(BG)}^{i,j}(H_G^*(X), \mathbb{Q}) \Rightarrow H^{i+j}(X).$$

Remark 1.4. Formality of the Borel construction of a G -space X does not necessarily imply formality of the action map $G \times X \rightarrow X$. For instance, the Borel construction of $SO(3)$ acting on S^2 is formal over \mathbb{Q} , but the action map is not formal, as it is just the Hopf map when evaluated at a given point in S^2 .

Remark 1.5. The terminology *equivariantly formal* is often used in the literature for something quite unrelated, namely the condition that the cohomology of the Borel construction $H^*(X_{hG})$ is free as a $H^*(BG)$ -module. This is equivalent to asking that the Leray-Serre spectral sequence of the Borel fibration collapses at the E_2 -page, and implies that the E_2 -page of the Eilenberg-Moore spectral sequence is concentrated in the column $i = 0$. Neither notion of equivariant formality implies the other. We believe that our terminology is more compatible with the usage in rational homotopy theory. For us a G -space is equivariantly formal if its rational homotopy type is a formal consequence of the equivariant cohomology algebra $H_G^*(X)$ as an algebra over $H^*(BG)$. This notion is also called *G -formality* in [Lil03] and *bundle formality* in [Scu04], where the various notions of equivariant formality are compared.

1.2. Leray-Serre spectral sequence for the Borel construction. We next recall the construction of the Leray-Serre spectral sequence from the ∞ -categorical point of view. Given a map of spaces $X \rightarrow Y$, there is an adjunction of ∞ -categories

$$f^* : \text{Fun}(Y, \mathcal{D}(\mathbb{Q})) \rightleftarrows \text{Fun}(X, \mathcal{D}(\mathbb{Q})) : f_*$$

where the functor f^* is precomposition with f and f_* is its right adjoint. If $p : X \rightarrow *$ is the unique map, then p_* is a lax monoidal functor and $p_*(\mathbb{Q})$ is thus a commutative algebra object in the ∞ -category $\mathcal{D}(\mathbb{Q})$ which is quasi-isomorphic to the singular cochains E_∞ -algebra $C^*(X, \mathbb{Q})$ and also to the commutative algebra $\mathcal{A}_{\text{pl}}(X)$.

Remark 1.6. If X is a reasonable space (paracompact, Hausdorff and locally contractible) the category $\text{Fun}(X, \mathcal{D}(\mathbb{Q}))$ can be modelled by the full subcategory of the ∞ -category of $\mathcal{D}(\mathbb{Q})$ -valued sheaves on X spanned by the locally constant sheaves. The functor p_* coincides with what is usually called Rp_* in the sheaf literature. In this case the multiplicative structure on $\text{Rp}_*\mathbb{Q}$ can be strictified to a strict commutative dg-algebra using the Thom-Whitney construction [NA87].

For a general map $f : X \rightarrow Y$, we denote by $p : X \rightarrow *$ and $q : Y \rightarrow *$ the unique maps and we construct a filtration of $p_*(\mathbb{Q}) = q_*(f_*(\mathbb{Q}))$ by $q_*(f_*\mathbb{Q}, \tau)$, namely, the image of the canonical filtration τ under q_* . The resulting spectral sequence is given by

$$E_1^{-i,j} = H^{j-i}(X, H^i(f_*\mathbb{Q})[-i]) = H^{j-2i}(Y, \mathcal{H}^i(F))$$

where F denotes the fiber of f and $\mathcal{H}^i(F)$ denotes the local system $H^i(F)$ equipped with its monodromy action by the fundamental groupoid of Y . The above E_1 -page is isomorphic to the page ${}^{\text{LS}}E_2^{j-2i,i}$ of the Leray-Serre spectral sequence. In fact the two spectral sequences coincide up to renumbering (see Example 1.4.8 of [Del71] or [Ant24, Proposition 9.2])

Now, we specialize to the case of a G -space X , where G is a compact Lie group. We denote by $\mathcal{A}_G(X)$ the filtered complex $q_*(f_*\mathbb{Q}, \tau)$ producing the Leray-Serre spectral sequence for the map $f : X_{hG} \rightarrow BG = *_hG$, where $q : BG \rightarrow *$. This defines a contravariant symmetric monoidal functor from G -spaces to filtered complexes over \mathbb{Q} whose associated spectral sequence gives

$$E_1^{-i,j}(\mathcal{A}_G(X)) \cong {}^{\text{LS}}E_2^{j-2i,i}(X_{hG}) = H^{j-2i}(BG, H^i(X)) \Rightarrow H_G^{j-i}(X).$$

1.3. The orthogonal group. We recall some very classical facts about the orthogonal group $O(n)$ [Bor67]. This is a compact Lie group. If n is even, $n = 2m$, then a maximal torus of $O(n)$ is given by the $2m \times 2m$ matrices that have m blocks of size 2×2 on the diagonal that are elements of $SO(2)$. If n is odd, $n = 2m + 1$, a maximal torus of $O(n)$ is given by the $(2m + 1) \times (2m + 1)$ matrices that have m blocks of size 2×2 on the diagonal that are elements of $SO(2)$ and where the remaining diagonal element is a 1.

In the even case $n = 2m$ the Weyl group is given by the semi-direct product $\Sigma_m \times C_2^m$. This can be realized as a subgroup of $O(2m) \subset \text{Aut}_{\mathbb{R}}(\mathbb{C}^m)$. The subgroup Σ_m acts on \mathbb{C}^m by permuting the factors while the subgroup C_2^m acts on \mathbb{C}^m by applying complex conjugation to some of the factors.

In the odd case $n = 2m + 1$, the Weyl group is given by $(\Sigma_m \times C_2^m) \times C_2$. This can be realized as a subgroup of $O(2m + 1) \subset \text{Aut}_{\mathbb{R}}(\mathbb{C}^m \oplus \mathbb{R})$ in which $(\Sigma_m \times C_2^m)$ acts on the \mathbb{C}^m factor as in the previous case and the extra copy of C_2 acts by the sign representation on \mathbb{R} and fixes \mathbb{C}^m .

If instead of $O(n)$ we want to work with $SO(n)$, the maximal torus is the same as for $O(n)$ and the Weyl group is a subgroup of the Weyl group of $O(n)$ of index 2. In the even case $n = 2m$ it is the subgroup of $\Sigma_m \times C_2^m$ consisting of elements $(\sigma, \epsilon_1, \dots, \epsilon_m)$ such that

$$\text{sign}(\sigma)\epsilon_1 \cdots \epsilon_m = 1.$$

In the odd case $n = 2m + 1$, it is the subgroup $(\Sigma_m \times C_2^m) \times C_2$ consisting of elements $((\sigma, \epsilon_1, \dots, \epsilon_m), \eta)$ such that

$$\text{sign}(\sigma)\epsilon_1 \cdots \epsilon_m \eta = 1.$$

1.4. Rational equivariant homotopy theory for the orthogonal group. Let G be a Lie group. Denote \mathbb{T} its maximal torus and \mathfrak{W} its Weyl group. Given a G -space X , we may form the Borel construction $X_{h\mathbb{T}}$ for its \mathbb{T} -action. This comes with a map to $B\mathbb{T}$ and this map is \mathfrak{W} -equivariant. The following proposition (see [AB84] for example) shows that equivariant rational homotopy type for actions of compact Lie groups can be reduced to the case of tori.

Proposition 1.7. *Let X be a G -space. The map*

$$\mathcal{A}_{\text{pl}}^*(X_{hG}) \rightarrow \mathcal{A}_{\text{pl}}^*(BG)$$

is naturally quasi-isomorphic to the map obtained by applying \mathfrak{W} fixed points to

$$\mathcal{A}_{\text{pl}}^*(X_{h\mathbb{T}}) \rightarrow \mathcal{A}_{\text{pl}}^*(B\mathbb{T})$$

Remark 1.8. Since \mathfrak{W} is a finite group and we are working rationally, the fixed point functor is exact and therefore does not need to be derived.

From the previous subsection and the above proposition we deduce that there is a natural quasi-isomorphism

$$\mathcal{A}_{\text{pl}}^*(X_{hO(n)}) \simeq (\mathcal{A}_{\text{pl}}^*(X_{hSO(n)}))^{C_2}$$

where C_2 is the quotient of the Weyl group of $O(n)$ by the Weyl group of $SO(n)$.

As an application of this proposition let us recall the computation of the cohomology of $BO(n)$ and $B SO(n)$.

Proposition 1.9. *Let $n \geq 2$ be an integer. Then*

- *If n is odd*

$$H^*(B SO(n)) \cong H^*(BO(n)) \cong \mathbb{Q}[p_1, p_2, \dots, p_{\lfloor n/2 \rfloor}]$$

with p_i of degree $4i$.

- *If n is even $H^*(BO(n)) \cong \mathbb{Q}[p_1, \dots, p_{n/2}]$ with p_i of degree $4i$.*
- *If n is even $H^*(B SO(n)) \cong \mathbb{Q}[p_1, \dots, p_{n/2-1}, e]$ with p_i of degree $4i$ and e of degree n .*

Moreover, for n even the map $H^(BO(n)) \rightarrow H^*(B SO(n))$ sends $p_{n/2}$ to e^2 (and the other generators to the generators of the same name).*

2. HOMOTOPY ORBITS OF THE LITTLE DISKS ∞ -OPERAD

Let \mathcal{D}_n denote the little n -dimensional disks operad. There is an action of $O(n)$ on \mathcal{D}_n by rotating disks. That is, each space $\mathcal{D}_n(k)$ has an $O(n)$ -action and the composition maps

$$\mathcal{D}_n(k) \times \mathcal{D}_n(m_1) \times \dots \times \mathcal{D}_n(m_k) \rightarrow \mathcal{D}_n(m_1 + \dots + m_k)$$

are $O(n)$ -equivariant, where the source is given the diagonal action. In order to make sense of the *homotopy orbits* of \mathcal{D}_n with respect to $O(n)$ it is a good idea to treat the space of colours of the operad in the same way as the space of operations. One way to do this, which is also robust in homotopical terms, is to take the dendroidal nerve of \mathcal{D}_n . This is the functor from category **Tree** (denoted Ω in [CM13]) to spaces which to a tree T assigns the product, over the vertices v of T with $|v|$ incoming edges, of $\mathcal{D}_n(|v|)$. For the tree η , with no vertices, we assign the point. We will make no notational distinction between the operad and its dendroidal nerve.

We denote by $f\mathcal{D}_n$ the framed little disks operad, it is a topological operad given by

$$f\mathcal{D}_n(k) = \text{Emb}(D^{\sqcup k}, D)$$

where D denotes a standard open n -disk and Emb denotes the space of smooth embeddings. In fact this operad comes in two flavors depending of whether we consider orientation preserving embeddings or arbitrary embeddings. We use the notation $f\mathcal{D}_n^+$ for the orientation preserving version and $f\mathcal{D}_n$ for the other one.

For $G = O(n)$ or $SO(n)$, we define the homotopy G -orbits of \mathcal{D}_n to be the dendroidal space $(\mathcal{D}_n)_G$ obtained by taking homotopy orbits objectwise. It is an ∞ -operad, in the following sense.

Proposition 2.1. *For $G = O(n)$ or $SO(n)$, the dendroidal space $(\mathcal{D}_n)_{hG}$ satisfies the Segal and completeness conditions.*

Proof. To shorten notation, we will write $Y := (\mathcal{D}_n)_{hG}$ for the duration of this proof. Let T be a tree given as the union of two trees R and S intersecting along an edge e . To verify the Segal condition, we must check that the map

$$Y(T) \rightarrow Y(R) \times_{Y(e)}^h Y(S)$$

is a weak equivalence. But this is clear since this map is fibered over BG and the fiber is the isomorphism $\mathcal{D}_n(T) \rightarrow \mathcal{D}_n(R) \times \mathcal{D}_n(S)$. Completeness is also immediate since the restriction of Y to linear trees is the homotopically constant simplicial space BG . \square

The following proposition shows that the homotopy orbits of \mathcal{D}_n is the ∞ -operad associated to the framed little n -disks operad. This is not new and, for a different model for ∞ -operads, follows from results of [Lur17] as explained in [HKK22, Proposition 2.2].

Proposition 2.2. *The Rezk completion of $f\mathcal{D}_n$ is weakly equivalent to the $O(n)$ homotopy orbits of \mathcal{D}_n . Likewise, the Rezk completion of $f\mathcal{D}_n^+$ is weakly equivalent to the $SO(n)$ homotopy orbits of \mathcal{D}_n .*

Proof. We write the proof in the $O(n)$ case. The other case is completely similar. In preparation for the proof, we will need to introduce some notation. Given a tree T , we write $e(T)$ for its set of edges and $\lambda(T) \subset e(T)$ for its subset of leaves (the root does not qualify as a leaf). The group $\text{map}(e(T), O(n))$ will be denoted $O(T)$.

We will write X as short for $f\mathcal{D}_n$. For a corolla C , the space $X(C)$ has a natural action of $O(C)$ by pre and post-composition. Namely, given a smooth embedding $f : \lambda(C) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and an element $g := (g_e) \in O(C)$, the element $g \cdot f$ is the embedding $\lambda(C) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ whose restriction to the component corresponding to $i \in \lambda(C)$ is $g_{e_0} f(i, -) g_i^{-1}$, where e_0 denotes the root. Correspondingly, for a general tree T the space $X(T) = \prod_v X(C_v)$ has an action of $O(T)$. (Note $e(T)$ is the union of $e(C_v)$ where v runs over the vertices of T .) These actions are compatible with respect to morphisms in Tree . Given a morphism $R \rightarrow T$ in Tree , we have a map $e(R) \rightarrow e(T)$ and a corresponding map $O(T) \rightarrow O(R)$. This endows $X(R)$ with an action of $O(T)$. Then, the induced map

$$X(T) \rightarrow X(R)$$

is $O(T)$ -equivariant. So we let Z be the dendroidal space whose value at a tree T is the space of homotopy orbits

$$Z(T) := X(T)_{hO(T)}.$$

Claim: Z is a complete Segal dendroidal space and the map $X \rightarrow Z$ is a Rezk completion.

Verifying the Segal condition is similar to the proposition above. Let T be a tree decomposed as a union of two trees R and S , intersecting along an edge u . We need to show that the canonical map

$$(1) \quad X(T)_{hO(T)} \rightarrow X(R)_{hO(R)} \times_{X(u)_{hO(u)}}^h X(S)_{hO(S)}$$

is a weak equivalence. This map fibers over

$$(2) \quad BO(T) \rightarrow BO(R) \times_{BO(u)}^h BO(S)$$

where $BO(T) \cong \text{map}(e(T), BO(n))$ and similarly for the other terms. In particular, the map (2) is a weak equivalence, as $e(T) = e(R) \cup_{e(u)} e(S)$. The fiber of the map (1) over the map (2)

$$X(T) \rightarrow X(R) \times_{X(e)}^h X(S)$$

which is a weak equivalence since X is Segal. Therefore (1) is a weak equivalence, and so Z is Segal. It is also complete, since the restriction of Z to linear trees is weakly equivalent to the constant simplicial space $BO(n)$. The canonical map $X \rightarrow Z$ is, when restricted to linear trees, the

completion map from the nerve of $O(n)$ to the constant simplicial space $BO(n)$. In particular, it is trivially essentially surjective. So to show that $X \rightarrow Z$ is a Rezk completion, it remains to show that it is fully faithful. That means showing, for every corolla C with n edges, that the square

$$\begin{array}{ccc} X(C) & \longrightarrow & Z(C) \\ \downarrow & & \downarrow \\ \prod_{e \in C} X(e) & \longrightarrow & \prod_{e \in C} Z(e) \end{array}$$

is homotopy cartesian. But this is tautologically true, since X evaluated on a tree with no vertices is a point, and the remaining terms form the homotopy fiber sequence $X(C) \rightarrow X(C)_{hO(C)} \rightarrow BO(C)$.

To conclude the proof, observe that the map $\mathcal{D}_n \rightarrow f\mathcal{D}_n \rightarrow Z$ factors through $(\mathcal{D}_n)_{hO(n)}$ and the resulting factorization $(\mathcal{D}_n)_{hO(n)} \rightarrow Z$ is a degreewise weak equivalence, completing the proof. \square

Remark 2.3. It is important to remember that taking a fibrant replacement in the model category of complete dendroidal spaces can change a dendroidal space quite significantly. For example, take G a topological group and view it as a topological operad concentrated in arity 1. Then $N_d(G)$ is the dendroidal space that sends the linear tree $[n]$ to G^n and any other tree to \emptyset . On the other hand, a fibrant replacement of this dendroidal space sends any linear tree to BG (and non-linear trees to \emptyset). In particular, formality of these two diagrams are very different conditions. In the first case, it means formality of the Hopf algebra $\mathcal{A}_{\text{pl}}(G)$ whereas in the second case it means formality of $\mathcal{A}_{\text{pl}}(BG)$. This explains the apparent discrepancy between our main result and the fact that the framed little disks operad is not formal in odd dimensions (see [Mor19] and [KW17]).

3. LITTLE DISKS AND SOME SPECIAL AUTOMORPHISMS

3.1. A special automorphism of \mathcal{D}_2 . Recall that the group $\text{GT}_{\mathbb{Q}}$ consists of pairs (λ, f) where $\lambda \in \mathbb{Q}^\times$ and f is an element of the Malcev completion of F_2 . These pairs are required to satisfy some equations (see [Dri91, Equation 4.3, 4.4 and 4.10]). In particular the pair $\tau = (-1, 1)$ is an element of $\text{GT}_{\mathbb{Q}}$.

Proposition 3.1. *Let $\psi \in \mathbb{Q}^\times$. There exists an element $\varphi \in \text{GT}_{\mathbb{Q}}$ whose cyclotomic character is ψ and which commutes with τ .*

Proof. To prove this proposition, we recall some facts about the theory of associators from the operadic point of view (see [Fre17], [BN98], see also [CRL24] for a survey). There is an operad PaB in the category of groupoids whose classifying space is a model for the little 2-disks operad. Applying Malcev completion aritywise yields an operad $PaB_{\mathbb{Q}}$ in the category of groupoids enriched in commutative coalgebras. There is another operad $PaCD_{\mathbb{Q}}$ in the category of groupoids enriched in commutative coalgebras which is the associated graded of $PaB_{\mathbb{Q}}$ for the filtration by powers of the augmentation ideal. We can extend the scalars to any commutative \mathbb{Q} -algebra R and produce PaB_R and $PaCD_R$. From this perspective an associator over R is a map of operads

$$PaB_R \rightarrow PaCD_R$$

which is the identity on the operad of objects (recall that these are operads in groupoids enriched in coalgebras). A 1-associator is an associator which is moreover the identity in arity 2. This can be very explicitly described as a power series $\Phi \in R\langle\langle X, Y \rangle\rangle$ which is group-like and satisfies a pentagon and hexagon identity. This description, in turn, comes from a presentation of the operad PaB (see [CRL24, Theorem 2.30]).

Similarly, we may introduce the group GT_R and GRT_R of automorphisms of PaB_R and $PaCD_R$ respectively inducing the identity on objects. These two groups are functorial in R and can be viewed as affine group schemes. Evaluation in arity 2 induces morphisms

$$\text{GT}_R \rightarrow R^\times \text{ and } \text{GRT}_R \rightarrow R^\times$$

that we call the cyclotomic characters. Conjugation with an associator over R induces an isomorphism of groups $\text{GT}_R \cong \text{GRT}_R$ which further commutes with the two cyclotomic characters.

Finally the operads PaB_R and $PaCD_R$ both admit an involution corresponding topologically to the complex conjugation action on the little 2-disks operad. An even associator over R is defined to be an isomorphism $PaB_R \rightarrow PaCD_R$ which is a 1-associator and further commutes with the involutions. This can be explicitly described as a power series Φ as above which further satisfies the equation

$$(3) \quad \Phi(-X, -Y) = \Phi(X, Y)$$

This fact can be seen as a consequence of the presentation of PaB . An associator is determined by where it sends the arity three generator μ of PaB (denoted Φ in [CRL24, Theorem 2.30]). This generator μ is invariant under the involution, so an associator is even if and only if it sends μ to an element of $PaCD_R$ which is invariant under the involution. If we translate this condition into the power series Φ as in [CRL24, Corollary 2.32], we see that the condition is exactly equation (3).

Finally, the main point is that Drinfel'd has shown [Dri91, Proposition 5.4] that rational even associators exist. Moreover, the group $\text{GRT}_{\mathbb{Q}}$ splits as a semi-direct product $\text{GRT}_{\mathbb{Q}} \cong \text{GRT}_{\mathbb{Q}}^1 \rtimes \mathbb{Q}^{\times}$. It follows that we can obtain the desired element of $\text{GT}_{\mathbb{Q}}$ by conjugating $(1, \psi) \in \text{GRT}_{\mathbb{Q}}$ with any rational even associator. \square

3.2. A special automorphism of \mathcal{D}_n . For X a space, we denote by $X_{\mathbb{Q}}$ the rationalization of X in the sense of Bousfield. The functor $X \mapsto X_{\mathbb{Q}}$ is symmetric monoidal in the ∞ -categorical sense. In particular, if \mathcal{P} is a topological operad that we think of as a dendroidal space satisfying Segal condition, we may form $\mathcal{P}_{\mathbb{Q}}$ by applying rationalization objectwise. We have an equivalence of functors from Tree to CDGA .

$$\mathcal{A}_{\text{pl}}(\mathcal{P}) \simeq \mathcal{A}_{\text{pl}}(\mathcal{P}_{\mathbb{Q}})$$

Proposition 3.2. *Let $\mu \in \mathbb{Q}^{\times}$, let $n \geq 3$. There is an automorphism φ of $(\mathcal{D}_n)_{\mathbb{Q}}$ such that*

- (1) *The induced action of φ^* on $H^i(\mathcal{D}_n)$ is by multiplication by $\mu^{i/2}$ if n is odd and by $\mu^{ni/2(n-1)}$ if n is even.*
- (2) *The automorphism φ induces an automorphism of the ∞ -operad $((\mathcal{D}_n)_{\mathbb{Q}})_{hO(n)}$.*
- (3) *The induced action of φ^* on $H^i(BO(n))$ is by multiplication by $\mu^{i/2}$.*

Before proving this proposition, let us make some comments. Note that the cohomology of \mathcal{D}_n is concentrated in degrees that are multiples of $n-1$, and the cohomology of $BO(n)$ is concentrated in even degrees, so all the exponents of μ that appear in the proposition are integers.

Condition (2) implies in particular that $BO(n)_{\mathbb{Q}}$ has an action of φ since this space is the value of $((\mathcal{D}_n)_{\mathbb{Q}})_{hO(n)}$ at linear trees. This is the action that we refer to in part (3). Moreover (2) is asserting that the action of φ on \mathcal{D}_{2n} is $O(n)$ -equivariant in a ‘‘twisted’’ sense. Precisely, it does not commute with the $O(n)$ action but commutes with it up to a non-trivial automorphism of $O(n)$.

Proof. We denote by $X \mapsto X^{\wedge}$ the pro-unipotent completion of a space. This is related to the rationalization in nice cases (see [BdBH21, Proposition 2.11]) but better behaved for our purposes in non-simply connected situations.

In order to construct automorphisms of $(\mathcal{D}_n)_{\mathbb{Q}}$, we use the following sequence of equivalences from [BdBH21]:

$$\text{Aut}((\mathcal{D}_n)_{\mathbb{Q}}) \rightarrow \text{Aut}((\mathcal{D}_n^{\wedge})_{\mathbb{Q}}) \rightarrow \text{Aut}((j^* \mathcal{D}_n^{\wedge})_{\mathbb{Q}}) \rightarrow \text{Aut}((j^* \mathcal{D}_2^{\wedge})_{\mathbb{Q}}^{\boxtimes \lfloor n/2 \rfloor} (\boxtimes j^* \mathcal{D}_1))$$

where in the last automorphism space the factor $j^*(\mathcal{D}_1)$ is present if n is odd and absent if n is even. The first of these maps is a weak equivalence by the fact that \mathcal{D}_n is simply connected [BdBH21, Proposition 2.11]. For the second map, j^* denotes the restriction functor induced by a certain functor $j : \text{simp}(\text{Fin}) \rightarrow \text{Tree}$ from the category of simplices of the nerve of Fin . It induces an

equivalence by [BdBH21, Proposition 6.3]. Finally the last map is an equivalence by the additivity theorem of [BdBW18] which remains true after completion by [BdBH21, Theorem 5.5].

Now, the operad $PaB_{\mathbb{Q}}$ in the previous proposition is a model for \mathcal{D}_2^{\wedge} , the pro-unipotent completion of the operad \mathcal{D}_2 . Let us write φ' for the automorphism of \mathcal{D}_2^{\wedge} corresponding to μ in Proposition 3.1. In the even case, consider the box product of $n/2$ -copies of φ' and in the odd case, consider the box product of $\lfloor n/2 \rfloor$ copies of φ' with one copy of the identity of \mathcal{D}_1 . In either case, this produces an automorphism of $(\mathcal{D}_n)_{\mathbb{Q}}$ thanks to the sequence of equivalences above. We call this automorphism φ and check that it satisfies the three conditions. For condition (1), this follows from the fact that this automorphism acts with degree $\mu^{\lfloor n/2 \rfloor}$ in arity 2 [BdBH21, Propositions 7.2 and 8.2].

It remains to prove condition (2) and (3). We first prove that the analogous conditions are satisfied with $O(n)$ replaced by its maximal torus \mathbb{T} (see subsection 1.3 for details about the maximal torus and its Weyl group). In the odd or even dimensional cases, the decomposition that we have chosen is compatible with the action of \mathbb{T} so (2) follows easily. Part (3) can be reduced the case $n = 2$ by additivity. But this case follows from the fact that

$$\mathrm{Aut}^h((\mathcal{D}_2)_{\mathbb{Q}}) \simeq \mathrm{GT}_{\mathbb{Q}} \times \mathrm{SO}(2)_{\mathbb{Q}}$$

where the semi-direct product structure is defined through the cyclotomic character. In particular our automorphism φ acts with degree μ on $\mathrm{SO}(2)_{\mathbb{Q}} = S_{\mathbb{Q}}^1$ which implies (3).

Now we prove (2) and (3) for the group $O(n)$. For this it suffices to use the equation

$$((\mathcal{D}_n)_{\mathbb{Q}})_{hO(n)} \simeq (((\mathcal{D}_n)_{\mathbb{Q}})_{h\mathbb{T}})_{h\mathfrak{W}}.$$

Therefore, we just need to prove that the action of φ on $((\mathcal{D}_n)_{\mathbb{Q}})_{h\mathbb{T}}$ is compatible with the \mathfrak{W} -action. This is clear for the symmetric group part of \mathfrak{W} (see subsection 1.3 for an explicit description of \mathfrak{W}). For the $(C_2)^{\lfloor n/2 \rfloor}$ subgroup, this comes from the fact that the automorphism φ' of $(\mathcal{D}_2)_{\mathbb{Q}}$ that we had chosen commutes with τ . \square

4. PURITY AND FILTERED FORMALITY

In this section we prove abstract results on filtered complexes, generalizing the results of E_r -formality of filtered commutative dg-algebras of [CG14] and of symmetric monoidal functors of [CH20] in the setting of mixed Hodge structures. The main idea is that, given a filtered commutative dg-algebra A together with an automorphism φ , then the purity conditions on the eigenvalues of φ^* acting on a certain page of its associated spectral sequence, give a quasi-isomorphism of commutative dg-algebras between A and the penultimate page of its associated spectral sequence, viewed as a commutative dg-algebra. We prove this functorially.

Throughout this section we fix a unit ξ in \mathbb{Q} different from ± 1 .

Definition 4.1. Let V be a $\mathbb{Q}[\varphi]$ -module V which is finite dimensional as a \mathbb{Q} -vector space and let i be an integer. We say that V is *pure of weight i* if either $V = 0$ or i is an integer and the only eigenvalue of φ is ξ^i .

Proposition 4.2. *Let V and V' be two complexes of $\mathbb{Q}[\varphi]$ -modules. Assume that, for all $k \geq 0$,*

- (1) $H^k(V)$ and $H^k(V')$ are of finite dimension, and
- (2) $H^k(V)$ is pure of weight i and $H^k(V')$ is pure of weight i' .

Then, if $i \neq i'$, we have $\mathrm{Map}_{\mathrm{D}(\mathbb{Q}[\varphi])}(V, V') \simeq 0$.

Proof. The ring $\mathbb{Q}[\varphi]$ is of homological dimension 1 so we may assume that V and V' have cohomology concentrated in a single degree which can even be assumed to be zero up to shifting. The modules V and V' then split as a direct sum of factors of the form $\mathbb{Q}[\varphi]/(f)$ in which the roots of f are ξ^i and $\xi^{i'}$ respectively. The result now follows from the general fact that for coprime elements f and g in a PID R , we always have $\mathrm{Map}_{\mathrm{D}(R)}(R/f, R/g) = 0$. \square

The following is an abstract statement of the fact that purity implies formality.

Theorem 4.3. *Let $\alpha \neq 0$ be a rational number. When restricted to cochain complexes A in $D(\mathbb{Q}[\varphi])$ whose cohomology $H^i(A)$ is pure of weight αi , there is an equivalence of strong symmetric monoidal ∞ -functors $H \simeq \text{Id}$, where $H : D(\mathbb{Q}[\varphi]) \rightarrow D(\mathbb{Q}[\varphi])$ is the cohomology functor.*

Although this result can be proven directly, we will give a proof in a more general setting of filtered complexes.

Denote by $D_r(\mathbf{F}R)$ the ∞ -category of filtered cochain complexes of R -modules (with non-negative increasing filtrations), where the weak equivalences are given by those maps of filtered complexes inducing an isomorphism at the E_{r+1} -page of the associated spectral sequences. This may be presented by the model structure obtained in [CELW20].

By Theorem 2.19 of [CG16], for all $r \geq 0$ there are symmetric monoidal equivalences of ∞ -categories

$$\text{Dec} : D_{r+1}(\mathbf{F}R) \longrightarrow D_r(\mathbf{F}R),$$

where Dec denotes Deligne's décalage functor, defined on a filtered complex (A, d, W) by

$$\text{Dec}W_i A^n := W_{i-n} A^n \cap d^{-1}(W_{i-n-1} A^{n+1}).$$

Note that the spectral sequences of a filtered complex and its décalage are related by a shift on their pages: there is a natural quasi-isomorphism of bigraded complexes

$$E_0^{-i,j}(\text{Dec}A) \longrightarrow E_1^{j-2i,i}(A)$$

inducing isomorphisms at the later pages (see [Del71]).

Let $\alpha \neq 0$ be a rational number. Denote $D_0^\alpha(\mathbf{F}\mathbb{Q}[\varphi])$ the full subcategory of $D_0(\mathbf{F}\mathbb{Q}[\varphi])$ spanned by those filtered complexes (A, W) such that:

- (i) For all $i \geq 0$, $H^*(Gr_i^W A)$ is finite-dimensional.
- (ii) If αi is an integer, then $E_1^{-i,j}(A)$ is pure of weight αi . Otherwise $E_1^{-i,j}(A) = 0$.

For $r \geq 0$, denote by $D_r^\alpha(\mathbf{F}\mathbb{Q}[\varphi])$ the full subcategory of $D_r(\mathbf{F}\mathbb{Q}[\varphi])$ spanned by objects A such that $\text{Dec}^r(A)$ lies in $D_0^\alpha(\mathbf{F}\mathbb{Q}[\varphi])$.

Remark 4.4. A sufficient condition for a filtered complex of $\mathbb{Q}[\varphi]$ -modules to lie in $D_r^\alpha(\mathbf{F}\mathbb{Q}[\varphi])$ is that $E_{r+1}^{-i,j}(A)$ is finite-dimensional and pure of weight $\alpha((1-r)i + jr)$ whenever this number is an integer, and zero otherwise. Of course this property is stable under turning pages in the spectral sequence so it suffices for it to be satisfied at some page E_ℓ with $\ell \leq r$.

Given a filtered complex A , then for all $r \geq 0$ the E_r -term of its associated spectral sequence may also be viewed as a filtered complex with filtration given by

$$W_p E_r(A)^n := \bigoplus_{i \leq p} E_r^{-i, n+i}(A).$$

In particular, we may view E_r as a symmetric monoidal endofunctor in $D_r(\mathbf{F}\mathbb{Q}[\varphi])$.

Theorem 4.5. *When restricted to $D_r^\alpha(\mathbf{F}\mathbb{Q}[\varphi])$, there is an equivalence of strong symmetric monoidal ∞ -functors $E_r \simeq \text{Id}$.*

Proof. We will proceed by induction over $r \geq 0$.

The base case of the induction is $r = 0$. In this case, consider the two functors

$$Gr : D_0^\alpha(\mathbf{F}\mathbb{Q}[\varphi]) \hookrightarrow D(\mathbb{Q}[\varphi])^\mathbb{N} : U$$

where Gr is the associated graded functor and U is the functor that takes a graded complex $\{A_k\}_{k \in \mathbb{N}}$ to the filtered complex

$$W_i U(A) := \bigoplus_{k \leq i} A_k$$

We claim that the functor Gr is fully faithful. Assuming this for the moment, the result can be deduced as follows. In order to prove that E_0 is equivalent to Id , it suffices to prove that $Gr \circ E_0$ is equivalent to Gr . The functor E_0 is simply the composite $U \circ Gr$, since $Gr \circ U \cong \text{Id}$, the result follows immediatly.

Now, we prove that Gr is fully faithful. Let $C = \{W_i C\}_{i \in \mathbb{N}}$ and $D = \{W_i D\}_{i \in \mathbb{N}}$ be two objects in $\mathbf{D}_0^\alpha(\mathbf{FQ}[\varphi])$. We wish to prove that the map

$$\text{Map}_{\mathbf{D}_0^\alpha(\mathbf{FQ}[\varphi])}(C, D) \rightarrow \prod_i \text{Map}_{\mathbf{D}(\mathbb{Q}[\varphi])}(Gr_i(C), Gr_i(D))$$

is an equivalence.

We view both the source and target as functors of C and D . Clearly, both the source and target preserve colimits in the C variable. We have an equivalence

$$C \simeq \text{colim}_i W_i C$$

where $W_i C$ is equipped with the induced filtration :

$$W_0 C \rightarrow \dots \rightarrow W_{i-1} C \rightarrow W_i C \xrightarrow{\text{id}} W_i C \xrightarrow{\text{id}} \dots$$

It follows that we can reduce the proof to the case where C has a finite filtration. Now, we observe that we have a cofiber sequence

$$W_{i-1} C \rightarrow W_i C \rightarrow Gr_i C$$

which can be viewed as a cofiber sequence of filtered objects in which $Gr_i C$ is equipped with the filtration

$$\begin{aligned} W_j(Gr_i C) &= Gr_i C \text{ if } i \leq j \\ &= 0 \text{ else.} \end{aligned}$$

From this observation, we see that we may reduce to the case in which C is such that $W_p C = 0$ for $p < i$ and $W_p C = Gr_i(C)$ for $p \geq i$. For this specific C , we see that the map that we are considering is simply

$$\text{Map}_{\mathbf{D}(\mathbb{Q}[\varphi])}(Gr_i C, W_i D) \rightarrow \text{Map}_{\mathbf{D}(\mathbb{Q}[\varphi])}(Gr_i C, Gr_i D)$$

In order to prove that this map is an equivalence, it is enough to prove that its fiber is zero. This fiber is simply the mapping space $\text{Map}_{\mathbf{D}(\mathbb{Q}[\varphi])}(Gr_i C, W_{i-1} D)$. By a similar inductive argument, we may reduce to proving that

$$\text{Map}_{\mathbf{D}(\mathbb{Q}[\varphi])}(Gr_i C, Gr_k D) = 0$$

for all $k < i$. This follows from the proposition 4.2 and finishes the proof in the case $r = 0$.

For $r > 0$, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{D}_r^\alpha(\mathbf{FQ}[\varphi]) & \xrightarrow{E_r} & \mathbf{D}_r^\alpha(\mathbf{FQ}[\varphi]) \\ \downarrow \text{Dec} & & \downarrow \text{Dec} \\ \mathbf{D}_{r-1}^\alpha(\mathbf{FQ}[\varphi]) & \xrightarrow{E_{r-1}} & \mathbf{D}_{r-1}^\alpha(\mathbf{FQ}[\varphi]) \end{array}$$

where Dec are equivalences of ∞ -categories and E_{r-1} is naturally equivalent to the identity by induction hypothesis. Therefore

$$E_r \simeq \text{Dec}^{-1} \circ E_{r-1} \circ \text{Dec} \simeq \text{Dec}^{-1} \circ \text{Id} \circ \text{Dec} \simeq \text{Id}. \quad \square$$

Proof of Theorem 4.3. Given a complex A , endow it with the canonical filtration τ . Then, the purity condition on $H^*(A)$ makes (A, τ) into an object in $\mathbf{D}_0^\alpha(\mathbf{FQ}[\varphi])$. Therefore $(E_0(A), d_0)$ is naturally quasi-isomorphic to A . But since τ is the canonical filtration, we have $E_0(A) \simeq H(A)$. \square

5. PRELUDE

Before proving the main theorem let us illustrate how the abstract machinery of Section 4 on purity and filtered formality may be applied to a couple of simple particular situations.

First, let us consider $\mathbb{C}P^n$ with its $G := PGL_{n+1}(\mathbb{C})$ -action. Then we have

$$E_1^{-i,j}(\mathcal{A}_G(\mathbb{C}P^n)) = H^{j-2i}(BG) \otimes H^i(\mathbb{C}P^n) \Rightarrow H_G^{j-i}(\mathbb{C}P^n).$$

Since both $\mathbb{C}P^n$ and G are defined over \mathbb{Z} , by usual étale cohomology arguments, after extending scalars to \mathbb{Q}_ℓ we can equip $\mathcal{A}_G(\mathbb{C}P^n)$ with a Frobenius action φ in such a way that $E_1^{-i,j}(\mathcal{A}_G(\mathbb{C}P^n))$ becomes pure of weight $j - i$. This gives that, for all i , $H_G^i(\mathbb{C}P^n; \mathbb{Q}_\ell)$ is pure of weight i . By Theorem 4.3 we obtain that $\mathbb{C}P^n$ is G -equivariantly formal over \mathbb{Q}_ℓ . By descent of formality for maps of commutative dg-algebras the same is true over \mathbb{Q} .

Remark 5.1. Note that even for $n = 1$, the Leray-Serre spectral sequence does not collapse multiplicatively. Indeed, in that case, the inclusion $U(1) \rightarrow PGL_2(\mathbb{C})$ given by

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

is a homotopy equivalence and the computation is the same as in Example 7.1.

A similar argument gives the following general result.

Theorem 5.2. *Let X be a smooth complex projective variety together with an action of an algebraic group G . Then X is G -equivariantly formal.*

To prove this, one just needs to enlarge the notion of weight by considering complexes A over $\mathbb{Q}_\ell[\varphi]$ such that the eigenvalues of φ^* on the cohomology $H^i(A)$ are Weil numbers of weight i . The splitting Theorem 4.3 works as well for this bigger category.

Remark 5.3. In [Lil03], it is shown that compact Kähler manifolds with holomorphic actions are equivariantly formal under the assumption that the Leray-Serre spectral sequence degenerates at E_2 . Note that our argument shows that this is always the case for smooth projective varieties. Also, [Scu07] proves formality for holomorphic torus actions on simply connected compact Kähler manifolds, in the genuine equivariant setting.

As another illustrative example, we consider configuration spaces of points in \mathbb{C}^n , together with their $GL_n(\mathbb{C})$ -action. Let us first recall their cohomology, which will also be useful in proving the main theorem.

Theorem 5.4 (Arnold, Cohen). *There is an isomorphism of rings*

$$H^*(\text{Conf}_k(\mathbb{R}^n)) \cong \mathbb{Q}[x_{ij}]/(x_{ij}^2, x_{ij} - (-1)^n x_{ji}, x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij})$$

where $1 \leq i, j \leq k$ and $i \neq j$ and the classes x_{ij} are of degree $n - 1$.

Theorem 5.5. *The space $\text{Conf}_k(\mathbb{C}^n)$ is $GL_n(\mathbb{C})$ -equivariantly formal over \mathbb{Q} .*

Proof. To lighten notation, we write $X = \text{Conf}_k(\mathbb{C}^n)$ and $G = GL_n(\mathbb{C})$. The action is algebraic so, using étale cohomology arguments, after extending scalars to \mathbb{Q}_ℓ we can equip both X and BG with a Frobenius action. In this case, $H^i(BG; \mathbb{Q}_\ell)$ is pure of weight i and $H^j(X; \mathbb{Q}_\ell)$ is pure of weight $2nj/(2n - 1)$ (see [CH22]). This gives that, over \mathbb{Q}_ℓ , the group

$$E_1^{-i,j}(\mathcal{A}_G(X)) = H^{j-2i}(BG; \mathbb{Q}_\ell) \otimes H^i(X; \mathbb{Q}_\ell)$$

is pure of weight $\frac{1}{2n-1}((2-n)i + 2nj)$. Therefore $\mathcal{A}_G(X) \otimes \mathbb{Q}_\ell$ lands in the category $D_{2n-1}^\alpha(\mathbf{F}\mathbb{Q}_\ell[\varphi])$, with $\alpha = \frac{1}{2n-1}$. By Theorem 4.5, we obtain that a model for $\mathcal{A}_{\text{pl}}(X_{hG}) \otimes \mathbb{Q}_\ell$ is given by the E_{2n-1} -page of $\mathcal{A}_G(X) \otimes \mathbb{Q}_\ell$, which is also the E_{2n} -page of the Leray-Serre spectral sequence. The differential of this page is determined by $d_{2n} : H^{2n-1}(X) \rightarrow H^{2n}(BG)$ since the cohomology of X is generated as a ring by $H^{2n-1}(X)$. The map d_{2n} sends the generators x_{ij} to c_n where $c_n \in H^{2n}(BG)$ is the top Chern class.

$$\text{LS } E_{2n} = \begin{array}{c} \bullet \\ | \\ 2n-1 \\ | \\ \text{---} \\ | \\ \bullet \quad \bullet \quad \cdots \quad \bullet \\ 2 \quad 4 \quad \cdots \quad 2n \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \nearrow d_{2n} \\ \\ \\ \\ \\ \end{array}$$

Now, to prove formality of $\mathcal{A}_{\text{pl}}(X_{hG}) \otimes \mathbb{Q}_\ell$ it suffices to build a quasi-isomorphism of commutative dg-algebras between $(\text{LS } E_{2n}, d_{2n})$ and its cohomology. The cohomology is given by

$$\text{LS } E_{2n+1} = (\text{Ker } d_{2n} \cap H^*(X)) \otimes H^*(BG)/(c_n).$$

The quotient $H^*(BG)/(c_n)$ is isomorphic to a subalgebra of $H^*(BG)$ so we obtain an inclusion

$$(\text{LS } E_{2n+1}, 0) \hookrightarrow (\text{LS } E_{2n}, d_{2n})$$

which is obviously a quasi-isomorphism. This proves G -equivariant formality of X over \mathbb{Q}_ℓ , which can then be descended to \mathbb{Q} . \square

Remark 5.6. The theorem above recovers the known fact that $\mathcal{M}_{0,k+1}$, the moduli space of genus 0 curves with $k+1$ marked points, is formal. Indeed, the Borel construction $\text{Conf}_k(\mathbb{C})_{hGL_1(\mathbb{C})}$ is equivalent to the quotient $\text{Conf}_k(\mathbb{C})/GL_1(\mathbb{C})$, since the action is free, and this quotient is equivalent to $\mathcal{M}_{0,k+1} = \text{Conf}_{k+1}(\mathbb{P}^1)/PGL_2(\mathbb{C})$.

Remark 5.7. The theorems above capture the essence of the proof of our main theorem. However, there are several reasons that made the strategy explained above insufficient for our purposes.

- (1) It only gives formality with \mathbb{Q}_ℓ -coefficients.
- (2) It is not clear that the Galois action will be compatible with operadic composition.
- (3) It only works in even dimensions.
- (4) In the even dimensional case, it gives $GL_n(\mathbb{C}) \simeq U(n)$ -equivariant formality instead of $O(2n)$ -formality.

We believe that the first three problems could be overcome. The first one is the very classical problem of descent of formality which is well-known to hold for individual commutative dg-algebras (see [HS79]) and may be easily extended to morphisms, but a proof of functorial descent of formality seems to be missing from the literature. To solve the second one, we would need to argue that the little disks operad is an algebro-geometric object so that the Galois action is compatible with operadic composition. This is exactly the content of the paper [BdBHK24]. The third problem could be solved by additivity as in the present paper. The last problem though seems very difficult to avoid as it is a classical fact that the only element of the absolute Galois group of \mathbb{Q} that commutes with complex conjugation is the identity so there does not seem to be possible to construct an automorphism of \mathcal{D}_n that satisfies the requirements of Proposition 3.2 using merely the absolute Galois group action.

6. PROOF OF THE THEOREM

Let G be $O(n)$ or $SO(n)$. From the previous sections we have a functor $T \mapsto \mathcal{A}_G(\mathcal{D}_n(T))$ from the category Tree to filtered commutative dg-algebras over \mathbb{Q} . This has an associated spectral sequence and we consider the functor given by its E_{n-1} -page. If n is odd, then this functor has trivial differential. The following proposition deals with the case when n is even.

Proposition 6.1. *The functor $T \mapsto E_{2n-1}(\mathcal{A}_G(\mathcal{D}_{2n}(T)))$ is formal, for G equal to $SO(2n)$ and $O(2n)$.*

Proof. We first consider the case when $G = SO(2n)$. The category Tree contains two types of objects: the linear trees and the non-linear trees. If T is linear, $\mathcal{D}_{2n}(T) \simeq *$ and the Leray-Serre spectral sequence is simply

$${}^{\text{LS}}E_2^{i,j} = H^i(BSO(2n)) \Rightarrow H^i(BSO(2n))$$

In this case, formality is trivial. If T is non-linear, the Leray-Serre spectral sequence is given by

$${}^{\text{LS}}E_2^{i,j} = H^i(BSO(2n)) \otimes H^j(\mathcal{D}_{2n}(T)) \Rightarrow H_{SO(2n)}^{i+j}(\mathcal{D}_{2n}(T))$$

By Proposition 5.4 the cohomology algebra $H^*(\mathcal{D}_{2n}(T))$ is generated by $H^{2n-1}(\mathcal{D}_{2n}(T))$ so by multiplicativity of the spectral sequence, the differential d_{2n} is determined by its restriction to $H^{2n-1}(\mathcal{D}_{2n}(T))$. This restriction sends all the algebra generators to the Euler class e in $H^{2n}(BSO(2n))$. The quotient $H^*(BSO(2n))/(e)$ is isomorphic to $H^*(BSO(2n-1))$ and can naturally be viewed as a subalgebra of $H^*(BSO(2n))$, the subalgebra generated by Pontryagin classes p_i for $i < n$. It follows that we have an injection

$$(*) \quad H^*(BSO(2n-1)) \otimes (\text{Ker } d_{2n} \cap H^*(\mathcal{D}_{2n}(T))) \hookrightarrow {}^{\text{LS}}E_{2n} = H^*(BSO(2n)) \otimes H^*(\mathcal{D}_{2n}(T))$$

which is a map of commutative dg-algebras. Moreover, it is straightforward to check that this map is a quasi-isomorphism.

Let us denote by $T \mapsto H(T)$ the assignment given by sending T to $H^*(BSO(2n))$ if T is linear or

$$H^*(BSO(2n-1)) \otimes (\text{Ker } d_{2n} \cap H^*(\mathcal{D}_{2n}(T)))$$

if T is not linear. We claim that this assembles into a functor on Tree. The functoriality is obvious for maps between linear trees and for maps between non-linear trees. Now, we observe that in Tree, there are no maps from non-linear trees to linear trees. So it simply remains to define the functoriality for maps from a linear tree to a non-linear tree. In that case, we have the map

$$H^*(BSO(2n)) \cong H^*(BSO(2n)) \otimes \mathbb{Q} \xrightarrow{\rho \otimes 1} H^*(BSO(2n-1)) \otimes (\text{Ker } d_{2n} \cap H^*(\mathcal{D}_{2n}(T)))$$

where ρ is the canonical map $H^*(BSO(2n)) \rightarrow H^*(BSO(2n-1))$.

We have just explained how to construct, for each T , a quasi-isomorphism

$$H(T) \rightarrow E_{2n-1}(\mathcal{A}_{SO(2n)}(\mathcal{D}_{2n}(T))).$$

One easily checks that this is a natural quasi-isomorphism. This proves formality of the commutative dg-algebra $E_{2n-1}(\mathcal{A}_{SO(2n)}(\mathcal{D}_{2n}(T)))$.

To prove formality for the algebra $E_{2n-1}(\mathcal{A}_{O(2n)}(\mathcal{D}_{2n}(T)))$, we use that there is a C_2 -action on $E_{2n-1}(\mathcal{A}_{SO(2n)}(\mathcal{D}_{2n}(T)))$ whose fixed points is precisely $E_{2n-1}(\mathcal{A}_{O(2n)}(\mathcal{D}_{2n}(T)))$. We construct a C_2 -action on $H(T)$ using the obvious action on $H^*(BSO(2n))$ if T is linear and the restriction of the C_2 -action of $E_{2n-1}(\mathcal{A}_{SO(2n)}(\mathcal{D}_{2n}(T)))$ along the inclusion $(*)$ when T is not linear. This C_2 -action is functorial for trees and the map $H(T) \rightarrow E_{2n-1}(\mathcal{A}_{SO(2n)}(\mathcal{D}_{2n}(T)))$ defined above is C_2 -equivariant. Therefore, taking C_2 -fixed points of this map gives functorial formality of $E_{2n-1}(\mathcal{A}_{O(2n)}(\mathcal{D}_{2n}(T)))$. \square

Theorem 6.2. *Let G be $O(n)$ or $SO(n)$. The little n -disks operad \mathcal{D}_n is G -equivariantly formal.*

Proof. Let ξ be a square in \mathbb{Q}^\times different from ± 1 . Let μ be a square root of ξ . Using this choice of μ in Proposition 3.2, we can promote the functor $\mathcal{A}_G(\mathcal{D}_n(T))$ to a functor from Tree to the category $\text{D}_0(\mathbf{FQ}[\varphi])$ of filtered complexes of $\mathbb{Q}[\varphi]$ -modules up to E_1 -isomorphisms. Composed with the forgetful functor to complexes of \mathbb{Q} -vector spaces we recover the functor

$$T \mapsto \mathcal{A}_{\text{pl}}(\mathcal{D}_n(T)_{hG}).$$

In the odd case, we get that $E_1^{-i,j}(\mathcal{A}_G(\mathcal{D}_n))$ is pure of weight $j - i$, so Theorem 4.3 implies formality.

In the even case, we get that $E_1^{-i,j}(\mathcal{A}_G(\mathcal{D}_n))$ is pure of weight $\frac{1}{n-1} [i(n-2) + j(n-1)]$ so this fits in Theorem 4.5 with $\alpha = 1/(n-1)$ and $r = n-1$. Therefore we get a functorial model of $\mathcal{A}_{\text{pl}}(\mathcal{D}_n(T)_{hG})$, given by the penultimate page of the Leray-Serre spectral sequence, together with its differential. The result now follows from Proposition 6.1. \square

For applications in manifold calculus, it is useful to understand the $O(n)$ -equivariant homotopy type of the map $\mathcal{D}_n \rightarrow \mathcal{D}_{n+c}$ induced by the standard embedding $\mathbb{R}^n \rightarrow \mathbb{R}^{n+c}$. In this case, we have the following variant of our main theorem.

Theorem 6.3. *Let $n \geq 3$ and $c \geq 2$. Let $G = O(n)$ or $G = SO(n)$. The map $\mathcal{D}_n \rightarrow \mathcal{D}_{n+c}$ induced by the standard inclusion of \mathbb{R}^n in \mathbb{R}^{n+c} is G -equivariantly formal.*

Proof. As in Proposition 3.2, for any two units μ and ν in \mathbb{Q} , we may construct an automorphism φ of the map $(\mathcal{D}_n)_{\mathbb{Q}} \rightarrow (\mathcal{D}_{n+c})_{\mathbb{Q}}$ satisfying the following conditions

- (1) The induced action of φ^* on $H^i(\mathcal{D}_n)$ is by multiplication by $\mu^{i/2}$ if n is odd and by $\mu^{ni/2(n-1)}$ if n is even.
- (2) The induced action on $H^i(\mathcal{D}_{n+c})$ is by multiplication by $(\mu^{(n-1)/2}\nu)^{i/(n+c-1)}$ if n is odd or $(\mu^{n/2}\nu)^{i/(n+c-1)}$ if n is even.
- (3) The automorphism φ induces an automorphism of the map of ∞ -operads

$$((\mathcal{D}_n)_{\mathbb{Q}})_{hO(n)} \rightarrow ((\mathcal{D}_{n+c})_{\mathbb{Q}})_{hO(n)}.$$

- (4) The induced action of φ^* on $H^i(BO(n))$ is by multiplication by $\mu^{i/2}$.

This is constructed by taking the box product of the automorphism that we construct in Proposition 3.2 and any automorphism of $(\mathcal{D}_c)_{\mathbb{Q}}$ that acts with degree ν in arity 2 (which exists by [BdBH21, Theorem 7.1]).

Now, take $\xi = \sqrt{\mu}$ and $\nu = \xi^c$ if n is odd or $\nu = \xi^{\frac{cn}{n-1}}$ if n is even (since we have freedom in μ , we can assume that these powers exist in \mathbb{Q}^\times). Then, in the even case, the map

$$\mathcal{A}_G(\mathcal{D}_{n+c}) \rightarrow \mathcal{A}_G(\mathcal{D}_n)$$

gets promoted to a map in $D_r^\alpha(\mathbf{FQ}[\varphi])$ with $\alpha = 1/(n-1)$ and $r = n-1$. In the odd case, it is a map in the category of cochain complexes satisfying the condition of Theorem 4.3. In either case, we get formality of the map. \square

7. EQUIVARIANT COHOMOLOGY RING OF CONFIGURATION SPACES

Let G be $SO(m)$ or $O(m)$. By Theorem 6.2, a model for the Borel construction of G acting on $\text{Conf}_\ell(\mathbb{R}^m)$ is given by the rational equivariant cohomology ring. In this section, we compute these cohomology rings. The cases m even or m odd behave very differently.

7.1. Examples in the case of two points. Before we explain the general case, we illustrate this in two examples, for two points and in low dimension m .

Example 7.1. Consider the action of S^1 on $\text{Conf}_2(\mathbb{R}^3)$. The E_2 -page splits as a tensor product $H^*(BS^1) \otimes H^*(S^2)$. For degree reasons there are no differentials in this spectral sequence. But the E_2 page is not multiplicatively isomorphic to the cohomology of the Borel construction, i.e. the Borel fibration is not rationally trivial.

To describe the equivariant cohomology ring, we observe that the action of S^1 on $\text{Conf}_2(\mathbb{R}^3) \simeq S^2$ is the suspension of the action of S^1 on itself, and so the Borel construction is the wedge $BS^1 \vee BS^1$. This gives the cohomology ring

$$H^*(\text{Conf}_2(\mathbb{R}^3)_{hS^1}) \cong \mathbb{Q}[a, b]/(ab)$$

where $|a| = |b| = 2$. The action of $\mathfrak{W} = C_2$ on $BS^1 \vee BS^1$ exchanges the summands, so exchanges a and b . (It corresponds to the degree one map on S^2 given by $(x, y, z) \mapsto (y, x, -z)$.) The homotopy fiber sequence

$$\mathrm{Conf}_2(\mathbb{R}^3) \rightarrow \mathrm{Conf}_2(\mathbb{R}^3)_{hS^1} \rightarrow BS^1$$

is equivalent to the homotopy fiber sequence $S^2 \rightarrow \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$, where the first map is the fold map $S^2 \rightarrow S^2 \vee S^2$ composed with the inclusion of the 2-skeleton. In cohomology, this sequence has the form $\mathbb{Q}[q] \rightarrow \mathbb{Q}[a, b]/(ab) \rightarrow \mathbb{Q}[x]/x^2$ where $q \mapsto a - b$ and $a, b \mapsto x$. Changing variables, $x = a$ and $q = a - b$, we have

$$H^*(\mathrm{Conf}_2(\mathbb{R}^3)_{hS^1}) \cong \mathbb{Q}[x, q]/x(x - q),$$

where $|x| = |q| = 2$. The corresponding C_2 -action on this algebra sends $x \mapsto x - q$ and $q \mapsto -q$. Setting $y = 2x - q$, we have $H^*(\mathrm{Conf}_2(\mathbb{R}^3)_{hS^1}) \cong \mathbb{Q}[y, q]/(y^2 - q^2)$. Now,

$$H^*(\mathrm{Conf}_2(\mathbb{R}^3)_{hSO(3)}) \cong H^*(\mathrm{Conf}_2(\mathbb{R}^3)_{hS^1})^{C_2}$$

and it follows that $H^*(\mathrm{Conf}_2(\mathbb{R}^3)_{hSO(3)}) \cong \mathbb{Q}[y, q^2]/(y^2 - q^2)$. This calculation is consistent with the fact that the $SO(3)$ -homotopy orbits of S^2 is equivalent to BS^1 .

As for the Σ_2 -action on $\mathrm{Conf}_2(\mathbb{R}^3)_{hS^1}$ which permutes labels, it sends $x \mapsto q - x$ and $q \mapsto q$. And for $\mathrm{Conf}_2(\mathbb{R}^3)_{hSO(3)}$ it sends y to $-y$.

Example 7.2. Now, consider the case of $\mathrm{Conf}_2(\mathbb{R}^4) \simeq S^3$ with its $O(4)$ -action. First, we restrict this action to the maximal torus $\mathbb{T} = S^1 \times S^1$. The Borel construction is then given by the wedge $BS^1 \vee BS^1$ as can be seen from the following \mathbb{T} -equivariant homotopy cocartesian square

$$\begin{array}{ccc} S^1 \times S^1 & \longrightarrow & S^1 \times D^2 \\ \downarrow & & \downarrow \\ D^2 \times S^1 & \longrightarrow & S^3 \end{array}$$

In this case the E_2 page of the Leray-Serre spectral sequence is given by

$$E_2^{*,*} = H^*(B\mathbb{T}) \otimes H^*(S^3) = \mathbb{Q}[q_1, q_2] \otimes \mathbb{Q}[x]/x^2$$

There is a unique differential d_4 determined by

$$d_4(x) = q_1 q_2$$

The E_5 -page is thus $\mathbb{Q}[q_1, q_2]/(q_1 q_2)$ which is indeed the cohomology of the wedge.

Finally we consider the action of $SO(4)$ and $O(4)$. The spectral sequences are the Weyl fixed points of the previous one. Here \mathfrak{W} is $\Sigma_2 \times C_2$ and $\Sigma_2 \times (C_2)^2$, respectively. For $SO(4)$, the Weyl fixed points of the E_2 -page can be identified with the subalgebra generated by $q_1^2 + q_2^2$, $q_1 q_2$ and x with differential $d_4(x) = q_1 q_2$. Therefore the E_5 -page is

$$E_5^{*,*} = \mathbb{Q}[q_1^2 + q_2^2].$$

For $O(4)$, the Weyl fixed points of the E_2 -page is $\mathbb{Q}[q_1^2 + q_2^2]$ and so in this case $d_4 \equiv 0$. In fact, the Borel construction relative to $O(4)$ and $SO(4)$ is in both cases rationally equivalent to $\mathbb{H}P^\infty$.

7.2. General case in even dimensions. In the even dimensional case $m = 2n$, the Leray-Serre spectral sequence does not degenerate at the E_2 -page, but the E_{2n+1} -page is multiplicatively isomorphic to the cohomology. For $G = SO(2n)$, we have

$$H_{SO(2n)}^*(\mathrm{Conf}_\ell(\mathbb{R}^{2n})) \cong H^*(BSO(2n - 1)) \otimes K,$$

where K is the kernel of the differential d_{2n} restricted to the zeroth-column of the Leray-Serre spectral sequence, i.e. $H^*(\mathrm{Conf}_\ell(\mathbb{R}^{2n}))$. It is determined by $d_{2n}(x_{ij}) = e$, where e is the Euler class.

For $G = O(2n)$ we have to take C_2 -fixed points of the above formula. This gives

$$H_{O(2n)}^*(\text{Conf}_\ell(\mathbb{R}^{2n})) \cong H^*(BSO(2n-1)) \otimes K^{C_2}.$$

The C_2 -fixed points of $H^*(\text{Conf}_\ell(\mathbb{R}^{2n}))$ are the elements of degree $2i(2n-1)$, $i \geq 1$. So K^{C_2} is simply the intersection of K with $\bigoplus_i H^{2i(2n-1)}(\text{Conf}_\ell(\mathbb{R}^{2n}))$.

7.3. General case in odd dimensions. In the odd dimensional case, the situation is harder. The Leray-Serre spectral sequence degenerates at the E_2 -page, but not multiplicatively. So we must argue differently. Although we take coefficients in \mathbb{Q} , the results in this section should apply if coefficients are taken in a ring where 2 is invertible.

Proposition 7.3. *The \mathbb{T}^n -equivariant cohomology of $\text{Conf}_\ell(\mathbb{R}^{2n+1})$ is isomorphic to the algebra generated by y_{ij} , $1 \leq i < j \leq \ell$, and q_1, \dots, q_n , with y_{ij} of degree $2n$ and q_u of degree 2, modulo the modified Arnol'd relation*

$$y_{ij}y_{jk} - y_{jk}y_{ik} - y_{ik}y_{ij} + p_n = 0$$

and the relation $y_{ij}^2 = p_n$, where $p_n = (q_1 \dots q_n)^2$.

Proof. We make a start by setting $\ell = 2$. The action of \mathbb{T}^n on $\text{Conf}_2(\mathbb{R}^{2n+1}) \simeq S^{2n}$ is the suspension of the action of \mathbb{T}^n on the equator S^{2n-1} . Therefore, the homotopy orbits of the \mathbb{T}^n -action on S^{2n} is identified with the pushout of

$$B\mathbb{T}^n \leftarrow (S^{2n-1})_{h\mathbb{T}^n} \rightarrow B\mathbb{T}^n$$

The cohomology of this pushout is the fiber product

$$P := H^*(B\mathbb{T}^n) \times_{H^*(S_{h\mathbb{T}^n}^{2n-1})} H^*(B\mathbb{T}^n).$$

We know $H^*(S_{h\mathbb{T}^n}^{2n-1}) \cong \mathbb{Q}[q_1, \dots, q_n]/(q_1 \dots q_n)$ (e.g. from proposition 6.1 or by noticing that $S_{h\mathbb{T}^n}^{2n-1}$ is the base change of $BSO(2n-1) \rightarrow BSO(2n)$ along the inclusion $B\mathbb{T}^n \rightarrow BSO(2n)$ and the Euler class $e \in H^{2n}(BSO(2n))$ is mapped to $q_1 \dots q_n$). And the two maps in the fiber product

$$\mathbb{Q}[q_1, \dots, q_n] \rightarrow \mathbb{Q}[q_1, \dots, q_n]/(q_1 \dots q_n)$$

send q_i to q_i and $-q_i$, respectively. (The difference in sign has to do with the fact that the action of \mathbb{T}^n on the top hemisphere of S^{2n} is the reflection of the action on the bottom hemisphere.) An element of P corresponds to a pair $(f^-, f + q_1 \dots q_n g)$ where f, f^- and g belong to $\mathbb{Q}[q_1, \dots, q_n]$ and $f^-(q_1, \dots, q_n) := f(-q_1, \dots, -q_n)$. Then we may define a map

$$P \rightarrow \mathbb{Q}[x, q_1, \dots, q_n]/(x^2 - xq_1 \dots q_n)$$

by sending $(f^-, f + q_1 \dots q_n g)$ to $f + xg$, where x is of degree $2n$. This is an isomorphism of $\mathbb{Q}[q_1, \dots, q_n]$ -algebras with inverse sending q_i to $(-q_i, q_i)$ and x to $(0, q_1 \dots q_n)$. Under the substitution $y := 2x - q_1 \dots q_n$, we get the presentation $\mathbb{Q}[y, q_1, \dots, q_n]/(y^2 - p_n^2)$ in the statement. This settles the case of two points.

Moving on, for an arbitrary ℓ and $i, j \in \{1, \dots, \ell\}$, let π_{ij} denote the projection

$$\text{Conf}_\ell(\mathbb{R}^{2n+1}) \rightarrow \text{Conf}_2(\mathbb{R}^{2n+1})$$

and consider the diagram with horizontal maps induced by the projections

$$\begin{array}{ccc} \bigotimes_{i < j} H^*(\text{Conf}_2(\mathbb{R}^{2n+1})_{h\mathbb{T}^n}) & \longrightarrow & H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n}) \\ \downarrow & & \downarrow \\ \bigotimes_{i < j} H^*(\text{Conf}_2(\mathbb{R}^{2n+1})) & \longrightarrow & H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})) \end{array}$$

The lower horizontal map is surjective (it enforces the Arnol'd relation). The left-hand map is identified with the tensor product, over i, j , of the map

$$\mathbb{Q}[x_{ij}, q_1, \dots, q_n]/(x_{ij}^2 - x_{ij}q_1 \dots q_n) \rightarrow \mathbb{Q}[x_{ij}]/x_{ij}^2.$$

which sends x_{ij} to x_{ij} and each q_u to zero (c.f. example 7.1). In terms of the generators $y_{ij} := 2x_{ij} - q_1 \dots q_n$, this is the map

$$\mathbb{Q}[y_{ij}, q_1, \dots, q_n]/(y_{ij}^2 - p_n^2) \rightarrow \mathbb{Q}[x_{ij}]/x_{ij}^2$$

which sends y_{ij} to $2x_{ij}$ and each q_u to zero. Now pick a basis of $H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1}))$ by monomials in the $2x_{ij}$, call it $\{b_u\}$. This basis lift to a preferred basis $\{b'_u\}$ of $H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$ by lifting it across the diagram above. The map

$$H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})) \otimes H^*(B\mathbb{T}^n) \rightarrow H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$$

which sends $b_u \otimes q$ to $b'_u q$ is an isomorphism of $H^*(B\mathbb{T}^n)$ -modules by the Leray-Hirsch theorem. (In particular, $2x_{ij}$ is sent to y_{ij} under this isomorphism.) The map

$$\mathbb{Q}[\{y_{ij}\}, q_1, \dots, q_n] \rightarrow H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$$

induced by the projection maps π_{ij} , respects the relation $y_{ij}^2 - p_n^2 = 0$ by construction. As for the Arnol'd relation, it is enough to verify it for three points since $y_{ij}y_{jk} - y_{jk}y_{ik} - y_{ik}y_{ij} + p_n$ is in the image of the map

$$H^*(\text{Conf}_3(\mathbb{R}^{2n+1})_{h\mathbb{T}^n}) \rightarrow H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$$

induced by forgetting all points but i, j, k . Now, the composition with

$$H^*(\text{Conf}_3(\mathbb{R}^{2n+1})_{h\mathbb{T}^n}) \rightarrow H^*(\text{Conf}_3(\mathbb{R}^{2n+1}))$$

satisfies the Arnol'd relation. This means we have relations of the form

$$(y_{ij}y_{jk} - y_{jk}y_{ik} - y_{ik}y_{ij}) + f_{ij}y_{ij} + f_{jk}y_{jk} + f_{ik}y_{ik} + g(q_1, \dots, q_n) = 0$$

in $H^*(\text{Conf}_3(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$ for some polynomials f_{ij}, f_{jk}, f_{ik} and g in the variables q_1, \dots, q_n . The claim is the polynomials f are zero and $g = p_n$. To see this, take a section s of p_{ij} which adds a far away point in $(0, \infty) \times \{0\} \subset \mathbb{R}^1 \times \mathbb{R}^{2n} = \mathbb{R}^{2n+1}$. By construction, s is \mathbb{T}^n -equivariant and so descends to a map

$$H^*(\text{Conf}_3(\mathbb{R}^{2n+1})_{h\mathbb{T}^n}) \rightarrow H^*(\text{Conf}_2(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$$

sending y_{ij} to y_{ij} , and sending y_{jk} and y_{ik} to $q_1 \dots q_n$. The image of our relation reads

$$y_{ij}q - q^2 - qy_{ij} + f_{ij}y_{ij} + f_{jk}q + f_{ik}q + g = 0$$

where q is short for $q_1 \dots q_n$. As such, f_{ij} must be zero. In the same way, by choosing an appropriate section of p_{jk} and p_{ik} , we conclude that f_{jk} and f_{ik} are also zero. And then we are left with the identity $g = q^2$.

Having described the map

$$\mathbb{Q}[\{y_{ij}\}, q_1, \dots, q_n]/\text{relations} \rightarrow H^*(\text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n})$$

what remains to do is showing that it is an isomorphism. By dimension counting, e.g. inducting up on length of the monomials in the y_{ij} , the two vector spaces have the same dimension. Finally, the map is surjective since every basis element b'_u is in the image. \square

Remark 7.4. Here is an illuminating graphical description of the algebra from proposition 7.3. The elements are linear combinations of graphs with set of vertices $\{1, \dots, \ell\}$ with coefficients in $\mathbb{Q}[q_1, \dots, q_n]$. These graphs are allowed to have multiple edges connecting vertices but no edges connecting a vertex to itself. However, the edges are not ordered, nor oriented. The relations on these graphs are of two kinds: the relation which says that a double edge may be removed at the expense of multiplying the resulting graph by $p_n = (q_1 \dots q_n)^2$, and the modified Arnol'd relation. Below is the graphical representation of the relations.

$$\begin{array}{c}
 \bullet \quad \bullet \\
 \text{\scriptsize } i \quad \text{\scriptsize } j \\
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array} = p_n \begin{array}{c} \bullet \quad \bullet \\ \text{\scriptsize } i \quad \text{\scriptsize } j \end{array}$$

$$\begin{array}{c}
 \bullet \quad \bullet \quad \bullet \\
 \text{\scriptsize } i \quad \text{\scriptsize } j \quad \text{\scriptsize } k \\
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array} - \begin{array}{c}
 \bullet \quad \bullet \quad \bullet \\
 \text{\scriptsize } i \quad \text{\scriptsize } j \quad \text{\scriptsize } k \\
 \text{---} \quad \text{---} \\
 \text{---} \quad \text{---}
 \end{array} = -p_n \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \text{\scriptsize } i \quad \text{\scriptsize } j \quad \text{\scriptsize } k \end{array}$$

The multiplication is given by superposition (or union) of graphs and multiplication of coefficients. The non-equivariant cohomology of $\text{Conf}_\ell(\mathbb{R}^{2n+1})$ (Theorem 5.4) has a similar, and well-known, graphical description. In cohomology, the map $\text{Conf}_\ell(\mathbb{R}^{2n+1}) \rightarrow \text{Conf}_\ell(\mathbb{R}^{2n+1})_{h\mathbb{T}^n}$ is given by

$$\begin{array}{c} \bullet \quad \bullet \\ \text{\scriptsize } i \quad \text{\scriptsize } j \end{array} \mapsto 2 \begin{array}{c} \bullet \quad \bullet \\ \text{\scriptsize } i \quad \text{\scriptsize } j \end{array}$$

The $SO(2n+1)$ -equivariant cohomology of $\text{Conf}_\ell(\mathbb{R}^{2n+1})$ is isomorphic to the \mathfrak{W} -fixed points of the \mathbb{T}^n -equivariant cohomology, where $\mathfrak{W} \subset (\Sigma_n \times C_2^n) \times C_2$ is the Weyl group as in section 1.3. The Weyl group acts on the coefficients $\mathbb{Q}[q_1, \dots, q_n]$ by permuting the q_i and changing sign. And it fixes the generators y_{ij} . (To see this, recall from the proof of proposition 7.3 that $y_{ij} = 2x_{ij} - q_1 \dots q_n$. An element $(\sigma, \epsilon_1, \dots, \epsilon_n, \eta) \in \mathfrak{W}$ acts as $q_1 \dots q_n \mapsto \eta q_1 \dots q_n$ (recall $\eta = \text{sign}(\sigma)\epsilon_1 \dots \epsilon_n$), acts as $x_{ij} \mapsto x_{ij} - q_1 \dots q_n$ if $\eta = -1$ and as $x_{ij} \mapsto x_{ij}$ if $\eta = 1$). So we deduce:

Proposition 7.5. *The $SO(2n+1)$ -equivariant cohomology of $\text{Conf}_\ell(\mathbb{R}^{2n+1})$ is isomorphic to the algebra generated by y_{ij} , $1 \leq i < j \leq \ell$ and p_1, \dots, p_n , with y_{ij} of degree $2n$ and p_u of degree $4u$, modulo the Arnold's relation*

$$y_{ij}y_{jk} - y_{jk}y_{ik} - y_{ik}y_{ij} + p_n = 0$$

and the relations $y_{ij}^2 = p_n$.

Finally, the $O(2n+1)$ -equivariant cohomology is the subalgebra generated by $y_{ij}y_{kl}$ with $i < j$ and $k < \ell$ and the Pontryagin classes p_u with $u \leq n$. As a vector space it is isomorphic to

$$H^*(BO(2n+1)) \otimes \left(\bigoplus_j H^{4jn}(\text{Conf}_\ell(\mathbb{R}^{2n+1})) \right).$$

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