HOMOTOPIE 2, FINAL EXAM
M2 MATHÉMATIQUES FONDAMENTALES , 2022–2023

Duration 3h.

Problem 1. Let $\mathtt{M}$ be a complete and cocomplete category. We assume that $\mathtt{M}$ has two model structures sharing the same cofibrations and fibrant objects. Show that the two model structure coincide. [Hint : first show that the two model structures have the same homotopy category].

Problem 2. Let R be a ring.

(1) Let $B \in \mathtt{Ch}_{\geq 0}(R)$. We consider $P(B)$ the chain complex given by
$$P(B)_n = \{(b, b'\gamma) \in B_n \oplus B_n \oplus B_{n+1}\}$$
with differential given by the formula
$$d(b, b', \gamma) = (db, db', d\gamma + b' - b)$$
Show that the map $B \to P(B)$ given by $b \mapsto (b, b, 0)$ and $P(B) \to B \times B$ given by $(b, b', \gamma) \mapsto (b, b')$ make $P(B)$ into a path object for B.

(2) Deduce from this an explicit construction of the homotopy pullback of a diagram $A \to B \leftarrow C$ in $\mathtt{Ch}_{\geq 0}(R)$.

(3) Recall that $S(n)$ denote the chain complex given by R in degree n and zero in any other degree. Let $n > 0$, show that the homotopy pullback of $0 \to S(n) \leftarrow 0$ is quasi-isomorphic to $S(n-1)$.

(4) Show that the pullback of $0 \to S(n) \leftarrow 0$ in the category $\mathtt{HoCh}_{\geq 0}(R)$ exists and is isomorphic to the zero chain complex.

(5) Deduce that for I the category $\bullet \to \bullet \leftarrow \bullet$, the canonical map
$$\mathtt{Ho}(\mathtt{Ch}_{\geq 0}(R)^I) \to (\mathtt{HoCh}_{\geq 0}(R))^I$$
is not an equivalence of categories.

Problem 3. We denote by $\mathtt{Gpd}$ the category of small groupoids. We denote by $[[n]]$ the category that has the same objects as $[n]$ and a unique morphism between any two object (observe that $[[n]]$ is groupoid).

(1) Show that for C a groupoid, there is a natural bijection
$$\mathrm{Hom}_{\mathtt{Cat}}([n], C) \cong \mathrm{Hom}_{\mathtt{Gpd}}([[n]], C)$$

(2) We denote by $\pi : \mathtt{sSet} \to \mathtt{Gpd}$ the unique colimit preserving functor whose restriction to the Yoneda's embedding $\Delta \subset \mathtt{sSet}$ is given by $[n] \mapsto [[n]]$. Show that there is an adjunction
$$\pi : \mathtt{sSet} \leftrightarrows \mathtt{Gpd} : N$$
where N is the nerve functor.

(3) We shall admit that the category $\mathtt{Gpd}$ is complete and cocomplete. Prove that there is a model structure on $\mathtt{Gpd}$ in which the cofibrations are the maps that are injective on objects, the weak equivalences are the equivalences of groupoids, the fibrations are the maps with the right lifting property against the map $[[0]] \to [[1]]$ sending 0 to 0.

(4) Show that the adjunction (π, N) is a Quillen adjunction (when we give $\mathtt{sSet}$ its usual model structure).

(5) Show that for C a groupoid, NC is a Kan complex. Show that $\pi_0(NC)$ is the set of isomorphisms classes of objects of C and that for x an object of C, $\pi_1(NC, x) \cong \mathrm{Hom}_C(x, x)$.

(6) Let G and H be two groups. We denote by $\mathcal{B}G$ and $\mathcal{B}H$ the groupoids with a unique object and G and H as morphisms. Show that there is an isomorphism
$$\mathrm{Hom}_{\mathtt{HoGpd}}(\mathcal{B}G, \mathcal{B}H) \cong \mathrm{Hom}_{\mathtt{Grp}}(G, H)/H$$
where H acts on the set of group homomorphisms $G \to H$ by the conjugation action. That is, the action of $h \in H$ on $f : G \to H$ is the morphism $g \mapsto h^{-1}f(g)h$.

(7) In the model category $\mathtt{Gpd}_*$ of pointed groupoids (i.e. groupoids with a map from [[0]]), show the following formula

$$\mathrm{Hom}_{\mathtt{HoGpd}_*}(\mathcal{B}G, \mathcal{B}H) \cong \mathrm{Hom}_{\mathtt{Grp}}(G, H)$$

Problem 4. (1) Let G a group and R a commutative ring. Let M be an $R[G]$-module. Recall that the group homology of G with coefficients in M is given by the formula

$$H_i(G; M) = \mathrm{Tor}_i^{R[G]}(R, M).$$

On the other hand, given an object X of a model category $\mathtt{M}$ equipped with a G-action, recall that we denote by X_{hG} the homotopy orbits of the G-action on X (homotopy colimit of the corresponding diagram). Show that there is an isomorphism

$$H_i(G; M) \cong H_i(M_{hG})$$

where M is viewed as chain complex of R-modules concentrated in degree 0.

(2) Using the left Quillen functor $N(-; R) : \mathtt{sSet} \to \mathtt{Ch}_{\geq 0}(R)$, show that there exists a simplicial set BG such that

$$H_*(G; R) \cong H_*(BG; R)$$

where on the left hand side, R is given the trival G-action.

(3) Now, we assume that G is a finite group and that $R = K$ is a field whose characteristic does not divide the cardinality of $|G|$. Show that the functor $V \mapsto V_G$ from $\mathtt{Mod}_{K[G]}$ to $\mathtt{Mod}_K$ is an exact functor.

(4) Deduce that if X is a simplicial set with a G-action, there is a natural isomorphism

$$H_*(X_{hG}; K) \cong H_*(X; K)_G$$

Problem 5. Recall that for K a based simplicial set and E a prespectrum, we define $K \wedge E$ to be the prespectrum defined by $(K \wedge E)_n = K \wedge E_n$ with obvious structure maps. Similarly, we denote by E^K the prespectrum defined by $(E^K)_n = \mathrm{map}_*(K, E_n)$ again with obvious structure maps.

(1) Show that $\mathtt{PSp}$ is a simplicial model category with tensor functor $\otimes : \mathtt{sSet} \times \mathtt{PSp} \to \mathtt{PSp}$ given by $K \otimes E = K_+ \wedge E$ where K_+ denotes $K \sqcup *$.

(2) Show that $\mathtt{Sp}$ is also a simplicial model category with the same tensor functor.

(3) Let $F : \mathtt{sSet} \to \mathtt{Sp}^{\mathrm{op}}$ be a functor that preserves homotopy colimits, show that F is naturally weakly equivalent to the functor $K \mapsto E^{K_+}$ for some spectrum E.

(4) Show that, for a spectrum E, the functor $K \mapsto \pi_i(E^{K_+})$ is naturally isomorphic to $K \mapsto E^{-i}(K)$ (the cohomology theory represented by E).

(5) Let us denote by $\mathbb{S}$ the sphere spectrum defined by $\mathbb{S} = \Sigma^\infty S^0$. For A a ring, let us denote by HA the Eilenberg-MacLane spectrum. Show that $\pi_0(\mathbb{S}) \cong \mathbb{Z}$ and that $\pi_0(HA) \cong A$.

(6) Construct a map $\mathbb{S} \to HA$ inducing the unit map $\mathbb{Z} \to A$ on π_0 and the zero map on any other homotopy group.

(7) Show that, if $i \neq 0$, $\pi_i(\mathbb{S}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$ [You can admit the following fact due to Serre : the homotopy groups of spheres are all finitely generated abelian groups, moreover, they are all finite except for $\pi_{4n-1}(S^{2n})$ for any $n > 0$]

(8) Show that the map $\mathbb{S} \to H\mathbb{Q}$ constructed in question (6) induces an isomorphism

$$\mathbb{S}^i(K) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^i(K; \mathbb{Q})$$

natural in the simplicial set K.