Homotopy II

M2 maths fondamentales Paris

Geoffroy Horel

CHAPTER 1

Model categories

1. Weak factorization systems

Definition 1.1. Let $\mathtt{C}$ be a category and $f : x \to y$ and $g : u \to v$ be two morphisms in $\mathtt{C}$. We say that g has the right lifting property against f or that f has the left lifting property against g if any commutative square of the following form

$$\begin{array}{ccc} x & \longrightarrow & u \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ y & \longrightarrow & v \end{array}$$

admits a lift, that is, there exists a morphism $h : y \to u$ such that both triangles in the following diagram commute

$$\begin{array}{ccc} x & \longrightarrow & u \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle g} \\ y & \longrightarrow & v \end{array}$$

More generally, if A and B are two classes of maps in $\mathtt{C}$, we say that the maps of A have the left lifting property against the maps of B (or that the maps of B have the right lifting property against the maps of A if for any map f of A and g of B, the map f has the left lifting property against g.

Notation 1.2. We shall write $f \perp g$ when f has the left lifting property agains g. We shall write $A \perp B$ if the class A has the left lifting property against B. We shall write $A^{\perp}$ for the class of maps that have the right lifting property against A and ${}^{\perp}A$ for the class of maps that have the left lifting property against A.

Proposition 1.3. *Let B be a class of morphisms in a category $\mathtt{C}$ and let $A =^{\perp} B$. Then*

(1) *Elements of A are stable under retracts.*
(2) *Elements of A are stable under pushouts.*
(3) *Elements of A are stable under composition.*
(4) *Elements of A are stable under transfinite composition.*

Proof. Exercise □

Remark 1.4. Of course this proposition admits a dual version. That is : maps with the right lifting property against maps of A are stable under retracts, pullbacks, composition.

Definition 1.5. Let $\mathtt{C}$ be a category. A pair (A, B) with A and B two classes of maps in $\mathtt{C}$ is called a weak factorization system if

(1) Any map f in $\mathtt{C}$ can be factored as $f = b \circ a$ with $a \in A$ and $b \in B$.
(2) We have $A \perp B$.
(3) The classes A and B are stable under retracts.

The following Lemma is called the retract argument and shows that in a weak factorization system, one class determines the other.

Lemma 1.6. *Let $\mathtt{C}$ be a category. Let (A, B) be a weak factorization system. Then $A = {}^{\perp}B$ and $B = A^{\perp}$.*

Proof. Let $p : x \to y$ be a map in $A^{\perp}$, we wish to prove that p is in B. The other case is proved dually. We factor p as

$$x \xrightarrow{a} z \xrightarrow{b} y$$

where a is in A and b is in B. So we have a commutative square

$$\begin{array}{ccc} x & \xrightarrow{\text{id}} & x \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} \\ z & \xrightarrow[b]{} & y \end{array}$$

A lift exists in this diagram and this gives us a map $s : z \to x$ such that the following diagram commutes.

$$\begin{array}{ccccc} x & \xrightarrow{a} & z & \xrightarrow{s} & x \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle p} \\ y & \xrightarrow[\text{id}]{} & y & \xrightarrow[\text{id}]{} & y \end{array}$$

This diagram exhibits p as a retract of a map in B so p must be in B as desired. □

2. The small object argument

The small objet argument is an efficient tool to produce weak factorization system with the additional property that the factorization is functorial.

Definition 1.7. A category I is called filtered if for any subcategory $J \subset I$ with J a category with finitely many morphisms, there exists a cocone for J in I.

Remark 1.8. In fact it can be shown that it is enough to check the following two conditions

- For any two objects x and y of I, there exists an object $z \in I$ and two maps $x \to z$ and $y \to z$.
- Fory any two maps $f_1, f_2 : x \to y$ in I, there exists a map $g : y \to z$ in I such that $gf_1 = gf_2$.

Example 1.9. The poset of natural numbers with the usual order is a filtered category. This will be our main example.

Definition 1.10. Let $\mathtt{C}$ be a category with filtered colimits. An object K of $\mathtt{C}$ is called compact if the functor

$$\operatorname{Hom}(K, -) : \mathtt{C} \to \mathtt{Set}$$

preserves filtered colimits.

Example 1.11. In the category of sets, the compact objects are the finite sets. In the category of vector spaces, the compact objects are the finite dimensional vector spaces.

Definition 1.12. Let $\mathtt{C}$ be a cocomplete category. Let I be a set of maps in $\mathtt{C}$. A relative I-cell complex is a map $f : X \to Y$ that can be factored as

$$X = X_0 \to X_1 \to X_2 \to \ldots \operatorname{colim}_{\mathbb{N}} X_n \cong Y$$

such that each map $X_i \to X_{i+1}$ is a pushout of a coproduct of maps of I.

Example 1.13. For example, if $I = \{S^n \to D^{n+1}, n \in \mathbb{N}\}$ in the category $\mathtt{Top}$, then a relative I-cell complex is just a relative cell complex in the usual sense.

Theorem 1.14. *Let $\mathtt{C}$ be a cocomplete category. Let I be a set of maps in $\mathtt{C}$ with compact sources. Then any map $f : X \to Y$ in $\mathtt{C}$ admits a functorial factorization of the form $f = p \circ i$ with i a relative I-cell complex and p a map with the right lifting property against the maps of I*

Proof. For $i : A_i \to B_i$ a map in I, let S_i be the set of commutative squares of the form

$$\begin{array}{ccc} A_i & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B_i & \longrightarrow & Y \end{array}$$

By the universal property of the coproduct, we can construct a commutative square

$$\begin{array}{ccc} \coprod_i \coprod_{S_i} A_i & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \coprod_i \coprod_{S_i} B_i & \longrightarrow & Y \end{array}$$

We define X_1 to be the pushout of the top horizontal and left vertical maps. By the universal property of the pushout there is a factorization of f as

$$X \to X_1 \to Y$$

Moreover, by construction, the map $X \to X_1$ is a relative I-cell complex. We repeat the same construction using the map $X_1 \to Y$. We obtain a factorization of f as

$$X \to X_1 \to X_2 \to Y$$

in which the first two maps are relative I-cell complexes. We can continue this process for each $n \in \mathbb{N}$ and we obtain a factorization

$$X \to X_\infty := \operatorname{colim}_n X_n \xrightarrow{p} Y$$

The first map is an I-cell complex and we shall prove that the map p is in $I^\perp$. Let us consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{g} & X_\infty \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

with $i \in I$. By compactness of A, the map g factors as

$$A \xrightarrow{g_n} X_n \to X_\infty$$

Moreover, by the universal property of the pushout and by construction of X_{n+1}, there is a lift in the commutative square

$$\begin{array}{ccccc} A & \xrightarrow{g_n} & X_n & \longrightarrow & X_{n+1} \\ {\scriptstyle i}\downarrow & & & & \downarrow \\ B & & \longrightarrow & & Y \end{array}$$

Composing this lift $B \to X_{n+1}$ with the canonical map $X_{n+1} \to X_\infty$, we get the desired lift in the initial square. □

Corollary 1.15. *Under the assumptions of the previous Theorem, the pair $({}^\perp(I^\perp), I^\perp)$ is a weak factorization system moreover, the maps in ${}^\perp(I^\perp)$ are exactly the retracts of the I-cell complexes*

Proof. Let us write $A = {}^\perp(I^\perp)$ and $B = I^\perp$, then clearly $A \perp B$ and A and B are closed under retracts. The factorization is obtained from the previous Theorem with the observation that an I-cell complex is a map in ${}^\perp(I^\perp)$. Moreover, let us denote by A' the retracts of the relative I-cell complexes. Then (A', B) also forms a weak factorization system by the retract argument. Therefore, we must have $A' = A$. □

3. Model categories

Definition 1.16. Let $\mathtt{C}$ be a category and W be a class of maps of $\mathtt{C}$ containing all isomorphisms. We say that W satisfy the two-out-of-six property if, given any triple

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$$

of composable morphisms of $\mathtt{C}$. If the composite $g \circ f$ and $h \circ g$ are in W, so are f, g, h and $h \circ g \circ f$.

Proposition 1.17. *Two-out-of-six implies two-out-of-three.*

Proof. Exercise □

Remark 1.18. One might wonder why working with the two-out-of-six property instead of the more naturel two-out-of-three property. At the end of the day, we will want to study the homotopy category of $\mathtt{C}$ with respect to W. Let us denote by W' the class of maps of $\mathtt{C}$ that are sent to isomorphisms by the localization functor $\mathtt{C} \to \mathtt{C}[W^{-1}]$. Then $W \subset W'$ and the canonical map $\mathtt{C}[W^{-1}] \to \mathtt{C}[W'^{-1}]$ is an equivalence. Moreover W' satisfies the two-out-of-six property thanks to the following proposition.

Proposition 1.19. *Let $F : \mathtt{C} \to \mathtt{D}$ be a functor. Let $Iso_{\mathtt{D}}$ be the class of isomorphisms of $\mathtt{D}$. Then $W = F^{-1}(Iso_{\mathtt{D}})$ satisfies the two-out-of-six property. More generally, if U is a class of map in $\mathtt{D}$ satisfying satisfies two-out-of-six, then $F^{-1}(U)$ also satisfies the two-out-of-six property.*

Proof. Exercise □

Definition 1.20. Let $\mathtt{C}$ be a category equipped with three classes of maps (W, C, F) whose elements are called respectively weak equivalences, cofibrations and fibrations. We say that this data forms a model category if the following axioms are satisfied.

(1) The category $\mathtt{C}$ has all limits and colimits.
(2) The class W satisfy the two-out-of-six property.
(3) The pairs $(W \cap C, F)$ and $(C, W \cap F)$ are weak factorization systems.

We shall use the terminology "trivial cofibration" for a map in $W \cap C$ and "trivial fibration" for a map in $X \cap F$. The terminology "acyclique (co)fibration" can also be found in the literature.

Remark 1.21. We see from the definition that any two of the three classes of maps determine the third.

Remark 1.22. Let us observe that the axioms are self-dual. That is, if $\mathtt{C}$ is a model category, then $\mathtt{C}^{\mathrm{op}}$ is also a model category with the same weak equivalences and with the cofibrarions of $\mathtt{C}^{\mathrm{op}}$ being the fibrations of $\mathtt{C}$ and the fibrations of $\mathtt{C}^{\mathrm{op}}$ being the cofibrations of $\mathtt{C}$.

Using the small object argument, we can construct model categories thanks to the following Proposition.

Proposition 1.23. *Let $\mathtt{C}$ be a complete and cocomplete category. Let I and J be two set of maps in $\mathtt{C}$ with compact source. Let W be a class of maps containing isomorphisms. Assume that*

(1) *W satisfies the two-out-of-six property.*
(2) *${}^{\perp}(I^{\perp}) \subset {}^{\perp}(I^{\perp}) \cap W$.*
(3) *$I^{\perp} \subset J^{\perp} \cap W$.*
(4) *Either $({}^{\perp}(I^{\perp})) \cap W \subset {}^{\perp}(J^{\perp})$ or $J^{\perp} \cap W \subset I^{\perp}$.*

Then there is a model structure (with functorial factorizations) on $\mathtt{C}$ with ${}^{\perp}(I^{\perp})$ as cofibrations, $J^{\perp}$ as fibrations and W as weak equivalences.

Proof. Exercise. □

Definition 1.24. A model category of this form is called a cofibrantly generated model category.

Remark 1.25. Observe that a cofibrantly generated model category has functorial factorizations. This is not part of the axioms of a model category but this is satisfied by most model categories of interests. For this reason we shall assume that all the model categories in this course have functorial factorizations.

Definition 1.26. Let $\mathtt{C}$ be a model category. Let $\varnothing$ be its initial object and $*$ its terminal object. We call an object X of $\mathtt{C}$ cofibrant if the map $\varnothing \to X$ is a cofibration. We call an object X fibrant if the map $X \to *$ is a fibration.

4. The homotopy category

Definition 1.27. Let $\mathtt{C}$ be a category and S be a class of morphisms of $\mathtt{C}$. The localization of $\mathtt{C}$ with respect to $\mathtt{S}$ is a category $L_S\mathtt{C}$ equipped with a functor $\mathtt{C} \to L_S\mathtt{C}$ such that for any category $\mathtt{D}$ the restriction functor

$$\mathrm{Fun}(L_S\mathtt{C}, \mathtt{D}) \to \mathrm{Fun}(\mathtt{C}, \mathtt{D})$$

is fully faithful with essential image the category of functors that sends maps of S to isomorphisms.

The category $L_S\mathtt{C}$ does not necessarily exist but if it does it is unique up to equivalence of categories. If one is willing to remove the hypothesis that the Hom sets are small, there is a construction of the localization as follows. Let us denote by $[1] = 0 \to 1$ the walking arrow category so that $\mathrm{Fun}([1], \mathtt{D})$ is the arrow category of $\mathtt{D}$. Similarly, let us denote by I the walking isomorphism category. This is the category with two objects and one isomorphisms between them. There is an obvious map $[1] \to I$. We can then construct $L_S\mathtt{C}$ via the following pushout diagram

$$\begin{array}{ccc} \sqcup_S[1] & \xrightarrow{f} & \mathtt{C} \\ \downarrow & & \downarrow \\ \sqcup_S I & \longrightarrow & L_S\mathtt{C} \end{array}$$

in which the map f sends the copy of the category $[1]$ indexed by $s \in S$ to the morphism s.

Remark 1.28. Note that with the explicit construction above, the category L_SC satisfies a stronger universal property. Namely, for $\mathtt{D}$ another category, the precomposition functor

$$\mathrm{Fun}(L_S\mathtt{C}, \mathtt{D}) \to \mathrm{Fun}(\mathtt{C}, \mathtt{D})$$

is a full subcategory (i.e. injective on objects and fully faithful) with image the functors that send maps of S to isomorphisms.

Definition 1.29. Let $\mathtt{C}$ be a model category. Let A be an object of $\mathtt{C}$. A cylinder object for A is a factorization of the fold map

$$A \sqcup A \to A$$

as

$$A \sqcup A \to A \otimes I \to A$$

where the first map is a cofibration and the second map is a weak equivalence.

A path object for A is a factorization of the diagonal map

$$A \to A \times A$$

as

$$A \to A^I \to A \times A$$

where the first map is a weak equivalence and the second map is a fibration.

Remark 1.30. These two notions are dual to each other (i.e. a path object in $\mathtt{C}$ is a cylinder object in $\mathtt{C}^{\mathrm{op}}$ and vice versa).

Remark 1.31. The notation A^I and $A \otimes I$ serve a psychological purpose but should not be interpreted as any form of tensor product or exponential object.

Definition 1.32. Consider two maps $f, g : A \to X$ in C. A left homotopy between this map is a map $h : A \otimes I \to X$ where $A \otimes I$ is a path object for A such that the composite

$$A \sqcup A \to A \otimes I \xrightarrow{h} X$$

coincides with (f, g).

One can define dually the notion of a right homotopy between f and g to be a a map $h : A \to X^I$ where X^I is a path object for X such that the composite

$$A \to X^I \to X \times X$$

coincides with (f, g).

Proposition 1.33. *Let A be a cofibrant object, then the left homotopy relation is an equivalence relation on* $\mathrm{Hom}_{\mathsf{C}}(A, X)$.

Proof. We prove the transitivity, the other properties are easier. Let $\alpha : C \to X$ be a homotopy between f and g and $\beta : D \to X$ be a left homotopy between g and h (note that the cylinders could be distinct so we write them differently). We denote by i_0 and i_1 the two maps $A \to C$ and j_0 and j_1 the two maps $A \to D$. One easily checks that they are cofibrations. One can then consider the following pushout square

$$\begin{array}{ccc} A & \xrightarrow{i_1} & C \\ {\scriptstyle j_0}\downarrow & & \downarrow \\ D & \longrightarrow & E \end{array}$$

By the universal property of the pushout, we obtain a map γ from E to X whose "restriction" to C is α and whose "restriction" to D is β.

We shall show that E is a cylinder object for A and that the map γ is a homotopy between f and h.

First of all, by the universal property of the pushout, we get a map $E \to A$ which is a weak equivalence. In order to construct a cofibration $A \sqcup A \to E$, we consider the following commutative diagram

$$\begin{array}{ccccc} & & A \sqcup A \sqcup A \sqcup A & \xrightarrow{(i_0,i_1)\sqcup(j_0,j_1)} & C \sqcup D \\ & & {\scriptstyle \mathrm{id}_A \sqcup (i_1, j_0) \sqcup \mathrm{id}_A}\downarrow & & \downarrow \\ A \sqcup \varnothing \sqcup A & \longrightarrow & A \sqcup A \sqcup A & \longrightarrow & E \end{array}$$

The square can easily be checked to be also a pushout square. Since the top map is a cofibration (as it is a coproduct of cofibrations) it follows that the bottom map is also a cofibration. The left map $A \sqcup A \to A \sqcup A \sqcup A$ is also a cofibration (as it is a coproduct of cofibrations) so the composite of the two bottom maps is a cofibration. Moreover, we check without difficulty that the composite $A \sqcup A \to E \to A$ coincides with the fold map. □

Proposition 1.34. *Let A be cofibrant and $p : X \to Y$ be a trivial fibration between fibrant objects, then the postcomposition by p map*

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$$

induces a bijection

$$\mathrm{Hom}(A, X)/\textit{left homotopy} \to \mathrm{Hom}(A, Y)/\textit{left homotopy}$$

Proof. The surjectivity simply follows from the lifting property (it is already true before passing to left homotopy classes).

Let us prove injectivity. Consider two maps $f : A \to X$ and $g : A \to X$ and let $h : A \otimes I \to Y$ be a left homotopy between the maps $p \circ f$ and $p \circ g : A$. Then, we consider the following commutative diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{(f,g)} & X \\ \downarrow & & \downarrow \\ A \otimes I & \xrightarrow[h]{} & Y \end{array}$$

Since we are in a model category, there exists a lift $A \otimes I \to X$ in this diagram. Such a lift is exactly a left homotopy between f and g. □

LEMMA 1.35 (Ken Brown's lemma). *Let $F : \mathtt{C} \to \mathtt{D}$ be a functor with $\mathtt{C}$ a model category and $\mathtt{D}$ a category equipped with a class of weak equivalences satisfying the two-out-of-three property. If F sends trivial cofibrations between cofibrant (resp. cofibrant-fibrant) objects to trivial cofibrations, then F sends all weak equivalences between cofibrant (resp. cofibrant-fibrant) objects to weak equivalences.*

Dually, if F sends trivial fibrations between fibrant objects to weak equivalences, then F sends weak equivalences between fibrant objects to weak equivalences.

PROOF. Let $f : A \to B$ be a weak equivalence between cofibrant (resp. cofibrant-fibrant) objects. Let us consider the pushout square

$$\begin{array}{ccc} \varnothing & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \sqcup B \end{array}$$

By assumption, the two maps $\varnothing \to A$ and $\varnothing \to B$ are cofibrations, it follows that all maps in this square are cofibrations. Let us consider the map $(f, \mathrm{id}_B) : A \sqcup B \to B$, we can factor this map as a cofibration followed by a trivial fibration

$$A \sqcup B \xrightarrow{u} C \xrightarrow{p} B$$

Let us now consider the following commutative diagram

$$\begin{array}{ccccc} A & & & & \\ \downarrow{\scriptstyle i_A} & & \searrow^{f} & & \\ A \sqcup B & \xrightarrow[u]{} & C & \xrightarrow[p]{} & B \\ \uparrow{\scriptstyle i_B} & & \nearrow_{\mathrm{id}_B} & & \\ B & & & & \end{array}$$

First, we observe that C is cofibrant (resp. cofibrant-fibrant) and that the map $u \circ i_B$ is a trivial cofibration (it is a cofibration as the composite of two cofibration and a weak equivalence by the two-out-of-three property since $p \circ u \circ i_B$ is a weak equivalence). It follows that $F(u \circ i_B)$ is a weak equivalence. By the same reasoning, $F(u \circ i_A)$ is a weak equivalence.

Now, the equation

$$F(p) \circ F(u \circ i_B) = F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$$

shows that $F(p)$ is a weak equivalence by the two-out-of-three property. Finally, we have

$$F(f) = F(p) \circ F(u \circ i_A)$$

so $F(f)$ is a weak equivalence as the composite of two weak equivalences. □

PROPOSITION 1.36. *Let $\mathtt{C}$ be a model category, let $\mathtt{C}_c$, $\mathtt{C}_f$ and $\mathtt{C}_{cf}$ denote respectively the full subcategory of cofibrant, fibrant and cofibrant-fibrant obejcts. Then, in the following* *diagram*

$$\begin{array}{ccc} \mathtt{HoC}_{cf} & \longrightarrow & \mathtt{HoC}_{c} \\ \downarrow & & \downarrow \\ \mathtt{HoC}_{f} & \longrightarrow & \mathtt{HoC} \end{array}$$

all the maps are equivalences of categories.

PROOF. We show it for the inclusion $\mathtt{C}_{cf} \to \mathtt{C}_f$. The proof is similar for all the other maps. We have a cofibrant replacement functor $Q : \mathtt{C} \to \mathtt{C}_c$. We observe that this functor sends fibrant objects to fibrant objects. Indeed, if Y is fibrant, we have a factorization of $\varnothing \to Y$ as

$$\varnothing \to Q(Y) \to Y$$

where the second map is a trivial fibration. It follows that the map $Q(Y) \to *$ is a fibration as the composite of two fibrations.

So we have a functor $Q : \mathtt{C}_f \to \mathtt{C}_{cf}$. This functor obviously preserve weak equivalences so it induces a functor $\mathtt{HoC}_f \to \mathtt{HoC}_{cf}$ which is easily seen to be an inverse of the functor induced by the inclusion. □

Let $\mathtt{C}$ be a model category. Let us denote by $\pi\mathtt{C}_{cf}$ the category whose objects are cofibrant-fibrant objects of $\mathtt{C}$ and the morphisms are left homotopy classes of maps (Exercise: check that composition is well-defined). There is a functor

$$\mathtt{C}_{cf} \to \pi\mathtt{C}_{cf}$$

THEOREM 1.37. *Let $\mathtt{C}$ be a model category. Then the projection functor*

$$\mathtt{C}_{cf} \to \pi\mathtt{C}_{cf}$$

induces an equivalence of categories

$$\mathtt{HoC}_{cf} \to \pi\mathtt{C}_{cf}$$

PROOF. We shall in fact prove that they are isomorphic categories using the unvirsal property of Remark 1.28. Both $\mathtt{HoC}_{cf}$ and $\pi\mathtt{C}_{cf}$ satisfy a universal properties as categories under $\mathtt{C}_{cf}$. In order to prove that they are isomorphic categories, we have to prove that their universal property are the same. In other words, we want to prove that, for a functor $F : \mathtt{C}_{cf} \to \mathtt{D}$ the following two properties are equivalent :

(1) If f is a weak equivalence, then $F(f)$ is an isomorphism.
(2) If f and g are two left homotopic maps, then $F(f) = F(g)$.

Assume that F satisfies (1), then consider a left homotopy between f and g :

$$\begin{array}{ccc} A & & \\ {\scriptstyle i_0}\downarrow & \searrow^{f} & \\ A \otimes I & \longrightarrow & X \\ {\scriptstyle i_1}\uparrow & \nearrow_{g} & \\ A & & \end{array}$$

from this diagram, we see that the map $F(f) \circ F(i_0)^{-1} = F(g) \circ F(i_1)^{-1}$ so the two maps $F(f)$ and $F(g)$ must be equal.

Now, we assume that F satisfies (2) and shall deduce that it satisfies (1). First, thanks to Ken Brown's lemma, it suffices to prove that F sends trivial fibrations to isomorphisms. So let $p : X \to Y$ be a trivial fibration. Then, according to Proposition 1.34, we see that the presheaves represented by X and Y on $\pi\mathtt{C}_{cf}$ are isomorphic through p. By Yoneda's lemma, it follows that $p : X \to Y$ is an isomorphism in $\pi\mathtt{C}_{cf}$. Thus F must send it to an isomorphism. □

This theorem implies that between cofibrant-fibrant objects, the left homotopy and right homotopy relation coincides. We can extend this theorem beyond cofibrant-fibrant objects thanks to the following.

PROPOSITION 1.38. *Let X be cofibrant and Y be fibrant. Then, the left homotopy relation coincides with the right homotopy relation on* $\mathrm{Hom}(X, Y)$ *moreover, the quotient by this relation is the Hom set from X to Y in* $\mathtt{HoC}$*.*

PROOF. We first show that the two relations coincide. Let $f, g : X \to Y$ be two left homotopic maps. Let $h : X \otimes I \to Y$ be a left homotopy and let us denote by i_0, i_1 the two maps $X \to X \otimes I$. Let $Y \to Y^I \to Y \times Y$ be a path object for Y. Then consider the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & Y^I \\ {\scriptstyle i_0}\big\downarrow & & & & \big\downarrow \\ X \otimes I & & \xrightarrow[(f\circ\pi,h)]{\qquad\qquad} & & Y \times Y \end{array}$$

in which π is the scond map in the cylinder $X \sqcup X \to X \otimes I \to X$. This diagram is commutative and a lift must exist since i_0 is a trivial cofibration. Let $k : X \otimes I \to Y \times Y$ be a lift. Then one checks easily that the composite $k \circ i_1$ is a right homotopy between f and g. We prove dually that right homotopic maps are left homotopic.

Now, we prove the second part of the proposition. Let us call the pair (X, Y) nice if the canonical map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)/\text{left homotopy}$$

coincides with the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\mathtt{HoC}}(X, Y)$$

We know from the previous theorem that this is true for X and Y cofibrant-fibrant. We have also seen that for $p : Y \to Y'$ a weak equivalence between fibrant object, then the induced map

$$\mathrm{Hom}(X, Y)/\text{left homotopy} \to \mathrm{Hom}(X, Y')/\text{left homotopy}$$

is a bijection (this follows from Proposition 1.34 for a trivial fibration and from Ken Brown's lemma in general). This shows that pairs (X, Y) with X cofibrant-fibrant and Y fibrant are nice. Dually if $X \to X'$ is weak equivalence between cofibrant objects and Y is fibrant, the induced map

$$\mathrm{Hom}(X', Y)/\text{right homotopy} \to \mathrm{Hom}(X, Y)/\text{right homotopy}$$

is a bijection. Since moreover the right homotopy relation coincides with the left homotopy relation, then we deduce that pairs (X, Y) with X cofibrant and Y fibrant are nice as desired. □

We can finally deduce the following fact from the above proof.

PROPOSITION 1.39. *Let X and Y be two objects of* $\mathtt{C}$ *with X cofibrant and Y fibrant. Then if two maps f and g are left homotopic with respect to one particular cylinder object $X \sqcup X \to X \otimes I \to X$, then they are left homotopic with respect to any choice of cylinder object.*

PROOF. The first paragraph of the previous proof shows that left homotopic implies right homotopic for any choice of path object and dually right homotopic implies left homotopic for any choice of cylinder. Composing these two implications, we get the desired result. □

5. Derived functors

Let $\mathtt{C}$ be a category equipped with a class of weak equivalences W satisfying the two-out-of-six property. Given a functor $F : \mathtt{C} \to \mathtt{D}$ sending maps of W to isomorphisms, there exists a unique extension of F into a functor $\mathtt{HoC} \to \mathtt{D}$. The problem of derived fucntors is to try to extend this to situations in which the functor F does not send all weak equivalences to isomorphisms.

Definition 1.40. Let $F : \mathtt{C} \to \mathtt{D}$ be a functor. We say that F is left derivable if there exists a functor $Q : \mathtt{C} \to \mathtt{C}$ together with a natural weak equivalence $q : Q \to \mathrm{id}_{\mathtt{C}}$ such that the restriction of F to the essential image of Q sends weak equivalences to isomorphisms.

Definition 1.41. We keep the notations of the previous definition. Assume that F is left derivable, then we define the left derived functor $\mathbb{L}F$ of F to be the composition

$$\mathtt{C} \xrightarrow{Q} \mathtt{C} \to \mathtt{D}$$

This functor sends weak equivalences in $\mathtt{C}$ to isomorphisms and we also denote by $\mathbb{L}F$ the induced functor

$$\mathtt{HoC} \to \mathtt{D}$$

This definition seems to depend a lot on the choice of Q. In fact it does not thanks to the following proposition and corollary.

Proposition 1.42. *Let $G : \mathtt{HoC} \to \mathtt{D}$ be a functor with a map $\alpha : G \to F$. Then, there is a unique factorization of α of the form*

$$G \xrightarrow{\alpha'} \mathbb{L}F \to F$$

in the category $\mathrm{Fun}(\mathtt{C}, \mathtt{D})$.

Proof. By definition, $\mathbb{L}F = F \circ Q$. We have a zig-zag

$$G \xleftarrow{Gq} G \circ Q \xrightarrow{\alpha Q} F \circ Q \xrightarrow{Fq} F$$

Since G is a homotopical functor, the left-pointing arrow is an isomorphism. Inverting this isomorphism we obtain the desired factorization.

Now assume that we have another factorization of α as

$$G \xrightarrow{\beta} F \circ Q \xrightarrow{F \circ q} F$$

Then we have a commutative diagram

$$\begin{array}{ccccc} GQ & \xrightarrow{\beta Q} & FQQ & \xrightarrow{FqQ} & FQ \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow[\beta]{} & FQ & \xrightarrow[Fq]{} & F \end{array}$$

In this diagram, the two vertical arrows of the left square are isomorphisms and the composite of the two top horizontal arrows is αQ. From this, we deduce that the preferred factorization constructed in the previous paragraph is equal to the composite $Fq \circ \beta$. □

Corollary 1.43. *Assume that F is left derivable, then the left derived functor $\mathbb{L}F$ does not depend on the choice of Q.*

Proof. Assume that we have two choices Q and Q' and let us denote by $\mathbb{L}F$ and $\mathbb{L}'F$ the two resulting derived functors. Then the natural transformation $\mathbb{L}F \to F$ must factor through $\mathbb{L}'F$ and similarly the natural transformation $\mathbb{L}'F \to F$ must factor through $\mathbb{L}F$ by the previous Proposition. Moerover, by uniqueness of factorizations, the two resulting maps $\mathbb{L}F \to \mathbb{L}'F$ and $\mathbb{L}'F \to \mathbb{L}F$ are mutually inverse isomorphisms. □

Proposition 1.44. *Same notations. Let X be an object of $\mathtt{C}$. Let $X_0 \to X$ be a weak equivalence whose source is in $Q(\mathtt{C})$. Then we have an isomorphism*

$$F(X_0) \cong \mathbb{L}F(X)$$

Similarly, let $f : X \to Y$ be a map in $\mathtt{C}$ and assume that we have a commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

where X_0 and Y_0 are in $Q(\mathtt{C})$ and the two vertical maps are weak equivalences, then we have an isomorphism between $\mathbb{L}F(f)$ and $F(f_0)$ in the arrow category of $\mathtt{D}$.

Proof. We leave the scond claim as an exercise. Let us prove the first claim. We have a commutative diagram

$$\begin{array}{ccc} QX_0 & \longrightarrow & QX \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

In this diagram, all maps but the top are weak equivalences by assumption. It follows from the two-out-of-three property that the top map must be a weak equivalence as well. Since F sends weak equivalences between objects of $Q(\mathtt{C})$ to isomorphisms, F must send the top map and the left map to isomorphisms. So we have a zig-zag of isomorphisms

$$F(X_0) \leftarrow F(QX_0) \to F(QX) := \mathbb{L}F(X)$$

as desired. □

Remark 1.45. The point of this proposition is that, once we know that a functor is left derivable, then, it is not necessary to use the functor Q to compute its left derived functor on an object X. We can use any choice of replacement of X by an object in $Q(\mathtt{C})$.

Example 1.46. Consider the category of chain complexes of left modules over a ring R. Let M be a right R-module. Then the functor

$$C \mapsto M \otimes_R C$$

is a functor from $\mathtt{Ch}_*(R)$ to $\mathtt{Ch}_*(\mathbb{Z})$. If M is a free R-module, this functor sends quasi-isomorphisms to quasi-isomorphisms, since in that case, we have a natural isomorphism

$$H_n(M \otimes_R C) \cong M \otimes_R H_n(C)$$

However in general tensoring with M is not an exact functor.

6. Quillen adjunction

Definition 1.47. Let $F : \mathtt{C} \to \mathtt{D}$ be a left adjoint functor between two model categories. Then F is called a left Quillen functor if F sends cofibrations to cofibrations and trivial cofibrations to trivial cofibrations.

Dually a right adjoint $G : \mathtt{C} \to \mathtt{D}$ is called a right Quillen functor if it sends fibrations to fibrations and trivial fibrations to trivial fibrations.

Proposition 1.48. *A left adjoint F is a left Quillen functor if and only if its right adjoint G is a right Quillen functor. In that case, we call the pair of adjoints (F, G) a Quillen adjunction.*

Proof. Exercise □

PROPOSITION 1.49. *Let $F : \mathtt{C} \to \mathtt{D}$ be a left Quillen functor. Then the composite*

$$F : \mathtt{C} \to \mathtt{D} \to \mathtt{HoD}$$

is left derivable.

Let $G : \mathtt{C} \to \mathtt{D}$ be a right Quillen functor. Then the composite

$$F : \mathtt{C} \to \mathtt{D} \to \mathtt{HoD}$$

is right derivable.

PROOF. Let $F : \mathtt{C} \to \mathtt{D}$ be a left Quillen functor. We take Q to be a cofibrant replacement functor. Indeed, by Ken Brown's lemma, F sends weak equivalences between cofibrant objects to weak equivalences. □

For a left Quillen functor F, we denote by $\mathbb{L}F$ the associated left derived functor and similarly for a right Quillen functor G, we denote by $\mathbb{R}G$ the associated right derived functor. We make a small abuse of notation and also denote by $\mathbb{L}F$ the functor $F \circ Q : \mathtt{C} \to \mathtt{D}$ where Q is a cofibrant replacement.

PROPOSITION 1.50. *Let $F : \mathtt{C} \leftrightarrows \mathtt{D} : G$ be a Quillen adjunction between model categories. Then the pair of functors*

$$\mathbb{L}F : \mathtt{HoC} \leftrightarrows \mathtt{HoD} : \mathbb{R}G$$

is an adjunction.

PROOF. Let A be a cofibrant object of $\mathtt{C}$ and X a fibrant object of $\mathtt{D}$. We denote by Q the cofibrant replacement fucntor in $\mathtt{C}$ and R the fibrant replacement functor in $\mathtt{D}$. Then we have a sequence of natural bijections

$$\begin{aligned}
[\mathbb{L}F(A), X] &\cong [F(QA), X] \\
&\cong \mathrm{Hom}(F(QA), X)/(\text{left homotopy}) \\
&\cong \mathrm{Hom}(QA, G(X))/(\text{left homotopy}) \\
&\cong [QA, G(X)] \\
&\cong [QA, G(RX)] \\
&\cong [QA, \mathbb{R}G(X)] \\
&\cong [A, \mathbb{R}G(X)]
\end{aligned}$$

□

DEFINITION 1.51. We say that a Quillen adjunction $F : \mathtt{C} \leftrightarrows \mathtt{D} : G$ is a Quillen equivalence if the following condition holds :

A map $F(A) \to X$ with A cofibrant and X fibrant is a weak equivalence if and only if the adjoint map $A \to G(X)$ is a weak equivalence.

PROPOSITION 1.52. *Let $F : \mathtt{C} \leftrightarrows \mathtt{D}$ be a Quillen equivalence. Then the functors $\mathbb{L}F$ and $\mathbb{R}G$ are mutually inverse equivalences of categories between $\mathtt{HoC}$ and $\mathtt{HoD}$.*

PROOF. We wish to show that the composite $\mathbb{L}F \circ \mathbb{R}G$ is naturally isomorphic to the identity of $\mathtt{HoD}$. We have a natural transformation

$$\mathbb{L}F(\mathbb{R}G(X)) := FQGR(X) \to F(GR(X)) \to R(X)$$

where the first map is induced by the natural transformation $Q \to \mathrm{id}_{\mathtt{C}}$ and the second map is the counit of the adjunction. The adjoint of this map is the map

$$QGR(X) \to GR(X)$$

induced by the natural transformation $Q \to \mathrm{id}$. This map is a weak equivalence with cofibrant source and fibrant target. It follows that our natrual transformation is a weak equivalence as desired.

We can dually construct a natural transformation $\mathrm{id}_{\mathtt{HoC}} \to \mathbb{R}G \circ \mathbb{L}F$ which is an isomorphism. □

This Proposition admits a converse.

Proposition 1.53. *Let $F : \mathtt{C} \leftrightarrows \mathtt{D}$ be a Quillen adjunction. Then it is a Quillen equivalence if the functors $\mathbb{L}F$ and $\mathbb{R}G$ are mutually inverse equivalences of categories between $\mathtt{HoC}$ and $\mathtt{HoD}$.*

Proof. Let $f : FA \to B$ be a weak equivalence with A cofibrant and B fibrant, we wish to show that the adjoint map $g : A \to GB$ is also a weak equivalence. Let us consider the following commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & GFA & \xrightarrow{G(f)} & GB \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \longrightarrow & GRFA & \xrightarrow[G(Rf)]{} & GRB \end{array}$$

Where the two unlabelled vertical maps are induced by the natural transformation $\mathrm{id} \to R$. Since B is fibrant, the right vertical map is a weak equivalence. Moreover, the map $A \to GRFA$ is a weak equivalence by assumption and the map $G(Rf)$ is a weak equivalence since it is obtained by applying G to a weak equivalence between fibrant objects. It follows that the composite of the two top horizontal maps is a weak equivalence as desired.

Similarly, if $g : A \to GB$ is a weak equivalence, its adjoint $f : FA \to B$ is a weak equivalence. □

CHAPTER 2

Examples of model structures

1. Chain complexes

We fix a commutative ring R. We denote by $\mathtt{Ch}_{\geq 0}(R)$ the category of chain complexes. An object of $\mathtt{Ch}_{\geq 0}(R)$ is given by a chain of R-modules :

$$\ldots \to C_3 \to C_2 \to C_1 \to C_0$$

with the composite of two consecutive maps equal to zero.

Definition 2.1. We say that a map $f : C \to D$ is a quasi-isomorphism if it induces an isomorphisms in homology in each degree.

We define some special objects of $\mathtt{Ch}_{\geq 0}(R)$. First $S(n)$ is the chain complex which is just R in degree n and zero everywhere else. Second $D(n)$ is the chain complex given by R in degree n and $n-1$ and zero everywhere else and with the only non-trivial differential being the identity map from R to R. There is a map $S(n) \to D(n+1)$ which is the identity map in degree n. We further define $S(-1)$ to be the zero chain complex and $D(0) = S(0)$.

We now define two families of maps I and J in $\mathtt{Ch}_{\geq 0}(R)$.

$$J = \{0 \to D(n), n \geq 1\}$$

and

$$I = \{S(n-1) \to D(n), n \in \mathbb{N}\}$$

Proposition 2.2. *A map has the right lifting property against the maps of J if and only if it is an surjective in each strictly positive degree.*

Proof. Giving a map $f : D(n) \to C_*$ is exactly the same as giving an element in C_n so having the right lifting property against the maps of J is exactly being in each strictly positive degree. □

Proposition 2.3. *A map has the right lifting property against the maps of I if and only if it is degreewise surjective and a quasi-isomorphism.*

Proof. Now, we study the right lifting property against the maps of I. A commutative square

$$\begin{array}{ccc} S(n-1) & \longrightarrow & C \\ \downarrow & & \downarrow p \\ D(n) & \longrightarrow & D \end{array}$$

is the data of an element y in D_n and a cycle $x \in C_{n-1}$ such that $px = dy$. A lift in this diagram is the data of an element x' in C_n such that $dx' = x$ and $px' = y$.

Now let us show that if $p : C \to D$ has the right lifting property against maps of I it is degreewise surjective. First observe that p is surjective on cycles. Indeed, if $y \in D_n$ is a cycle, then we can construct a diagram as above with $x = 0$ and a lift in this diagram gives us a cycle x' with $px' = y$. Now, we do not assume anymore that y is a cycle, then dy is a cycle so $dy = px$ for some cycle x and we can find a commutative square as above and a lift in this diagram gives us an element x' in C_n such that $px' = y$.

Now let us show that if $p : C \to D$ has the right lifting property against maps of I it is a quasi-isomorphism. The fact that it is surjective on cycles implies that it induces a surjective map in homology. Now, let $x \in C_{n-1}$ be an $(n-1)$-cycle such that px is a boundary, then picking $y \in D_n$ such that $px = dy$, we get a commutative square as above and the existence of a lift for this square shows that x is itself a boundary so p is injective in homology.

Finally, let us show that if $p : C \to D$ is degreewise surjective and a quasi-isomorphism it has the rigth lifting property against the maps of I. Consider a commutative square as above. Since p is surjective, we can find x'' such that $px'' = y$. The problem is that we may not have $dx'' = x$.

However, we have $pdx'' = dy = px$ so $x - dx''$ is sent to zero by p, moreover $x - dx''$ is a cycle. Since p induces an isomorphism in homology, we must have $x - dx'' = dx'''$. Moreover, we may pick x''' in the kernel of p (indeed this kernel is an acyclic complex). Now consider $x' = x'' + x'''$, then $dx' = x$ as desired and $px' = px'' + px''' = x$ as desired. □

THEOREM 2.4. *There is a model structure on* $\mathtt{Ch}_*(R)$ *in which the weak equivalences are the quasi-isomorphisms, the fibrations are the epimorphisms and the cofibrations are the maps that are degreewise injective with projective cokernel*

PROOF. The category is complete and cocomplete. The class of weak equivalences satisfies the two-out-of-six property.

We define C to be ${}^{\perp}(W \cap F)$ (we don't have a choice). We leave it as an exercise to check that the maps of C are exactly as described in the statement of the Theorem.

By the small object argument and the previous proposition, the pair $(C, W \cap F)$ is a weak factorization system generated by the set I. Similarly the pair $({}^{\perp}(J^{\perp}), F)$ is a weak factorization system generated by J. In order to conclude the proof, it suffices to prove that ${}^{\perp}(J^{\perp})$ coincides with $C \cap W$. Since $F \cap W \subset F$, we see that ${}^{\perp}(J^{\perp}) \subset C$. Clearly the maps of J are weak equivalences and in ${}^{\perp}(J^{\perp})$ so they must be in $C \cap W$. It follows that ${}^{\perp}(J^{\perp}) \subset C \cap W$. In order to prove the converse, we need to show that the maps of $C \cap W$ have the left lifting property against fibrations (Exercise). □

1.1. Cylinders. There is a nice construction of cylinder objects in $\mathtt{Ch}_{\geq 0}(R)$. Let C_* be a cofibrant chain complex. Then the cylinder on C is defined as

$$(C \otimes I)_n = C_n \oplus C_n \oplus C_{n-1}$$

with differential given by

$$d(c, c'k) = (dc - k, dc' + k, -dk)$$

Observe that $C \oplus C$ sits as a subcomplex of $C \otimes I$ (as the triples whose last component is zero). Moreover, there is a map

$$p : C \otimes I \to C$$

given by $p(c, c', k) \mapsto c + c'$. It is easy to check that the map $C \oplus C \to C \otimes I$ is a cofibration and that he map $C \otimes I \to C$ is degreewise surjective. It is also easy to check that the map

$$C \oplus C \to C \otimes I \to C$$

is the fold map. It remains to check the following.

PROPOSITION 2.5. *The map* $p : C \otimes I \to C$ *is a quasi-isomorphism.*

PROOF. First observe that if c is cycle of C, then $c = p(c, 0, 0)$ and $(c, 0, 0)$ is a cycle in $C \otimes I$. This shows that the map p induces a surjection in homology. Now, let (c, c', k) be a cycle in $C \otimes I$ (i.e. $dc = k = -dc'$) such that $c + c'$ is a boundary (i.e. $c + c' = dx$ for some x). Then we have

$$d(x, 0, c') = (c + c' - c', 0 + c', k) = (c, c', k)$$

which shows that (c, c', k) is a boundary and hence, p is injective in homology. □

REMARK 2.6. In that case the notation $C \otimes I$ is really a tensor product with I the cellular complex of the interval :

$$I = Re_0 \oplus Re_1 \leftarrow R\gamma \leftarrow 0 \leftarrow 0 \leftarrow \ldots$$

with differential $d\gamma = e_1 - e_0$.

Then we have the following easy proposition which says that algebraic homotopies in the usual sense are really left homotopies in the model structure $\mathtt{Ch}_{\geq 0}(R)$.

PROPOSITION 2.7. *Two maps $f, g : C \to D$ are left homotopic with respect to $C \otimes I$, if and only if there exists a sequence of maps $h_n : C_n \to D_{n+1}$ such that*

$$dh_n(x) = f(x) - g(x)$$

for all $x \in C_n$.

From this proposition, one deduce easily the following corollary.

COROLLARY 2.8. *Let C be any chain complex. Then, we have a natural isomorphism*

$$[S(n), C] \cong H_n(C)$$

PROOF. Maps $S(n) \to C$ are in one-to-one correspondance with cycles in C and it is easy to see that the two such maps are homotopic if and only if the corresponding cycles differ by a boundary. □

2. Constructing new model categories from old ones

In this section, we give three tools to construct model categories.

PROPOSITION 2.9. *Let $\{\mathtt{M}_i\}_{i\in I}$ be a family of model categories. Then there is a model structure on the product category $\prod_i \mathtt{M}_i$ in which a map is a weak equivalence, cofibration or fibration if all of its components are weak equivalences, cofibrations and fibrations.*

PROOF. Easy. □

PROPOSITION 2.10. *Let $\mathtt{M}$ be a model category and let X be an object of $\mathtt{M}$. Then there is a model structure on $\mathtt{M}/X$ in which a map is a weak equivalence, cofibration or fibration if and only if it is sent to one via the forgetful functor $\mathtt{M}/X \to \mathtt{M}$. Moreover the forgetful functor is a left Quillen functor.*

Similarly there is a model structure on $X/\mathtt{M}$ in which a map is a weak equivalence, cofibration or fibration if and only if it is sent to one via the forgetful functor $X/\mathtt{M} \to \mathtt{M}$. Moreover the forgetful functor is a right Quillen functor.

PROOF. Easy. □

Let $F : \mathtt{C} \leftrightarrows \mathtt{D} : G$ be an adjunction. Assume that $\mathtt{C}$ is a cofibrantly generated category with I as a generating set of cofibrations and J as a generating set of trivial cofibrations. Then we may try to define a model structure on $\mathtt{D}$ by declaring a map to be a weak equivalence (resp. fibration) if its image by G is a weak equivalence (resp. fibration). This model structure does not necessarily exists but, if it does, it is unique as the cofibrations have to be the maps with the left lifting property against trivial fibrations. Moreover, by construction the adjunction (F, G) is a Quillen adjunction. If the transferred model structure exists, playing with adjunctions, we see that $FI^{\perp}$ is exactly the set of trivial fibrations in the transferred model structure and $FJ^{\perp}$ is the set of fibrations.

THEOREM 2.11. *We assume that G preserves filtered colimits. Assume that relative FJ-cell complexes are weak equivalences. Then the transferred model structure exists.*

PROOF. The fact that G preserves filtered colimits implies that F preserves compact objects so we can apply the small object argument to deduce that $({}^{\perp}(FI^{\perp}), FI^{\perp})$ and $({}^{\perp}(FJ^{\perp}), FJ^{\perp})$ are weak factorization systems. It remains to chack that ${}^{\perp}(FJ^{\perp})$ is the class of trivial cofibrations. First, we observe that

$$Fib_{\mathtt{D}} \cap Weq_{\mathtt{D}} = FI^{\perp} \subset FJ^{\perp} = Fib_{\mathtt{D}}$$

this implies that

$${}^{\perp}(FJ^{\perp}) \subset {}^{\perp}(FI^{\perp}) := Cof_{\mathtt{D}}$$

Moreover, we know that relative FJ-cell complexes are weak equivalences. It follows that retracts of FJ-cell complexes are also weak equivalences and thus, by the small object argument, ${}^{\perp}(FJ^{\perp})$ consists of weak equivalences so ${}^{\perp}(FJ^{\perp}) \subset Cof_{\mathtt{D}} \cap Weq_{\mathtt{D}}$. Conversely, let $f \in Cof_{\mathtt{D}} \cap Weq_{\mathtt{D}}$, then by the small object argument, we can factor f as $p \circ i$ with i a relative FJ-cell complex and p a map in $FJ^{\perp}$. By the two-out-of three property p is a trivial fibration. So f has the left lifting property against p. Then we can argue as in Lemma 1.6 to show that f is a retract of p. So f must be in ${}^{\perp}(FJ^{\perp})$ □

Remark 2.12. If we further assume that the weak equivalences in $\mathtt{C}$ are stable under $\mathbb{N}$-indexed transfinite composition (which often happens), it suffices to check that pushouts of maps of FJ are weak equivalences. Indeed, if that's the case, relative FJ-cell complexes will be weak equivalences.

3. Simplicial sets

3.1. Ends and coends. Let I be a small category, let $\mathtt{C}$ be a cocomplete category. For S a set and C an object of $\mathtt{C}$, we make the convention that $S \times C$ denotes the coproduct of copies of C indexed by S. Let $F : I^{\mathrm{op}} \to \mathtt{Set}$ be a functor and $G : I \to \mathtt{C}$ be another functor. We define their coend denoted $F \otimes_I G$ as the following coequalizer

$$\bigsqcup_{f:i\to j} F(j) \times G(i) \rightrightarrows \bigsqcup_{i\in I} F(i) \times G(i) \to F \otimes_I G$$

In this coequalizer diagram, the top map restricted to the summand indexed by $f : i \to j$ is given by

$$F(j) \times G(i) \xrightarrow{F(f)\times \mathrm{id}} F(i) \times G(i)$$

and the bottom map restricted to the summand indexed by $f : i \to j$ is given by

$$F(j) \times G(i) \xrightarrow{\mathrm{id}\times G(f)} F(j) \times G(j)$$

Since colimits commute with colimits, it is easy to check that the assignment

$$(F, G) \mapsto F \otimes_I G$$

preserve colimits in both variables.

The following fact is often called the co-Yoneda lemma.

Proposition 2.13. *Let I be an object of I, let $F : I \to \mathtt{C}$ be any functor, then the coend*

$$\mathrm{Hom}_I(-, i) \otimes_I F$$

is isomorphic to $F(i)$.

Remark 2.14. This is in fact a particular case of a more general construction. Assume that we have a functor $H : I \times I^{\mathrm{op}} \to \mathtt{D}$, then the coend of H denoted $\int^I H$ is defined by the following coequalizer

$$\bigsqcup_{f:i\to j} H(j, i) \rightrightarrows \bigsqcup_{i\in I} H(i, i) \to \int^I H$$

The case above corresponds to $H(i, j) = F(i) \times G(j)$.

Another example of this construction is to take two covariant functors $F : I \to \mathtt{C}$ and $G : I \to \mathtt{C}$ and to consider the functor $H : I^{\mathrm{op}} \times I \to \mathtt{Set}$ given by $H(i, j) = \mathrm{Hom}(F(i), G(j))$. Then the coend of H^{op} (which is by definition the end of H) is the following equalizer in sets

$$\prod_{i\in I} \mathrm{Hom}(F(i), G(i)) \rightrightarrows \prod_{f:i\to j} \mathrm{Hom}(F(i), G(j))$$

which is nothing but the set of natural transformations from F to G.

3.2. Category of simplices and the density theorem.

DEFINITION 2.15. Let X be a simplicial set. The category of simplices of X denoted $\mathrm{Simp}(X)$ is the category whose objects are pairs $([n],\sigma)$ with $[n]\in\Delta$ and $\sigma\in X_n$ and morphisms from $([m],\sigma)$ to $([n],\tau)$ are maps $f:[m]\to[n]$ such that $X(f)\tau=\sigma$.

There is an obvious functor from $R_X:\mathrm{Simp}(X)\to\mathtt{sSet}$ sending $([n],\sigma)$ to $\Delta[n]$. Moreover, we observe that X is a cocone for this functor, in which the map

$$\Delta[n]\to X$$

corresponding to the object $([n],\sigma)$ is the classifying map for σ. By the universal property of the colimit, we obtain a map

$$\mathrm{colim}_{\mathrm{Simp}(X)} R_X\to X$$

PROPOSITION 2.16. *This map is an isomorphism.*

PROOF. By definition, the k-simplices of this colimit are given by the colimit

$$\mathrm{colim}_{([n],\sigma)\in\mathrm{Simp}(X)}\Delta[n]_k$$

which can be identified with the coend

$$\mathrm{Hom}_\Delta(-,[k])\otimes_\Delta X$$

which is exactly X_k by Yoneda's lemma. □

REMARK 2.17. This statement holds for any presehaf category and is called the density theorem. It gives an expression of any presheaf as a canonical colimit of representable presheaves.

3.3. Functors out of presehaves categories. We write this subsection for the category of simplicial sets but it works for any presheaf category. We have the Yoneda embedding

$$\Delta\to\mathtt{sSet}:=\mathrm{Fun}(\Delta^{\mathrm{op}},\mathtt{Set})$$

THEOREM 2.18. *Let* $\mathtt{C}$ *be a category with all colimits, let* $\mathrm{Fun}^L(\mathtt{sSet},\mathtt{C})\subset\mathrm{Fun}(\mathtt{sSet},\mathtt{C})$ *be the category of functors preserving all colimits. Then the restriction along the Yoneda embedding*

$$\mathrm{Fun}^L(\mathtt{sSet},\mathtt{C})\to\mathrm{Fun}(\Delta,\mathtt{C})$$

is an equivalence of categories.

PROOF. We shall only sketch the proof. We construct a functor that goes backward. Let $F:\Delta\to\mathtt{C}$ be a functor, we define $L(F)$ to be the functor $\mathtt{sSet}\to\mathtt{C}$ given by

$$LF(X):=X\otimes_\Delta F$$

We observe that the functor L and R form an adjunction

$$L:\mathrm{Fun}(\Delta,\mathtt{C})\leftrightarrows\mathrm{Fun}(\mathtt{sSet},\mathtt{C}):R$$

The counit of this adjunction $F\to R(L(F))$ is easily seen to be an isomorphism, so the functor L is fully faithful. Moreover, by the observation of the previous section, the functor $L(F)$ preserves colimits, it follows that the essential image of L consists of colimit preserving functors. Now, let F be a colimit preserving functor $\mathtt{sSet}\to\mathtt{C}$, the counit of the adjunction above gives us a natural transformation

$$LRF\to F$$

This natural transformation induces an isomorphism on the image of the Yoneda embedding. moreover, both the source and the target functor preserve colimits. Let X be a simplicial set and consider the functor $R_X:\mathrm{Simp}(X)\to\mathtt{sSet}$ of the previous subsection, we have a commutative square

$$\begin{array}{ccc}\mathrm{colim}\,LRF(R_X) & \longrightarrow & \mathrm{colim}\,F(R_X)\\ \downarrow & & \downarrow\\ LRF(X) & \longrightarrow & F(X)\end{array}$$

Since both LRF and F preserve colimits, the two vertical maps are isomorphisms and by the observation that LRF and F coincide on the image of the Yoneda embedding, we see that the top horizontal map is an isomorphism. It follows that the bottom map is an isomorphism as required.

□

EXAMPLE 2.19. There is a cosimplicial topological space $[n] \mapsto \Delta^n$ with Δ^n the standard n-simplex. By the previous theorem, there is a unique colimit preserving functor $\mathtt{sSet} \to \mathtt{Top}$ denoted $X \mapsto |X|$ such that the functor $[n] \mapsto |\Delta[n]|$ coincides with this cosimplicial space. The right adjoint to this functor is the singular simplicial set

$$S_\bullet(X) : [n] \mapsto S_n(X) := \mathrm{Hom}_{\mathtt{Top}}(\Delta^n, X)$$

EXAMPLE 2.20. The left adjoint to the nerve functor $\mathtt{Cat} \to \mathtt{sSet}$ is the unique colimit preserving functor from $\mathtt{sSet}$ to $\mathtt{Cat}$ that sends $\Delta[n]$ to $[n]$.

3.4. The internal Hom. The category of simplicial set has what is called an internal hom functor. That is, for any simplicial sets X and Y, there exists a simplicial set $\mathrm{map}(X, Y)$ with the following properties :

(1) The assignment $(X, Y) \mapsto \mathrm{map}(X, Y)$ is covariantly functorial in Y and contravariantly functorial in X.
(2) The zero simplices of $\mathrm{map}(X, Y)$ are in natural bijection with $\mathrm{Hom}(X, Y)$.
(3) There are well-defined composition maps

$$\mathrm{map}(X, Y) \times \mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$$

that satisfy the standard unitality and associativity properties.
(4) For any three simplicial sets X, Y, Z, there is a natural bijection

$$\mathrm{Hom}(X, \mathrm{map}(Y, Z)) \cong \mathrm{Hom}(X \times Y, Z)$$

Now, we come to the construction of $\mathrm{map}(X, Y)$. From Property (4) above, we see that we have

$$\mathrm{Hom}(\Delta[n], \mathrm{map}(X, Y)) \cong \mathrm{Hom}(\Delta[n] \times X, Y)$$

so we have

$$\mathrm{map}(X, Y)_n = \mathrm{Hom}(\Delta[n] \times X, Y)$$

and this indeed defines a simplicial set that satisfies all the properties above.

3.5. The nerve functor. There is a very important functor from categories to simplicial sets called the nerve functor. To construct it we first construct a cosimplicial object in $\mathtt{Cat}$ sending the ordered set $[n]$ to the category $[n]$ (recall that an ordered set is a category). So for a category C, we define

$$N(C)_n := \mathrm{Hom}_{\mathtt{Cat}}([n], C)$$

and this assignment is a contravariant functor from Δ to sets, i.e. a simplicial set. This functor is a right adjoint and its left adjoint is the unique functor from simplicial sets to categories that preserves colimits and that restricts to the functor

$$\Delta[n] \mapsto [n]$$

on the subcategory $\Delta \subset \mathtt{sSet}$.

3.6. The model structure. We denote by $\partial\Delta[n]$ the subsimplical set of $\Delta[n]$ defined by

$$\Delta[n]_i = \{f : [i] \to [n], f \text{ is not surjective}\}$$

In $\Delta[n]$ there is a unique non-degenerate n-simplex given by the identity map $[n] \to [n]$. We observe that $\partial\Delta[n]$ is the smallest subsimplicial set that contains all the faces of the non-degenerate n-simplex. There is a nice description of $\partial\Delta[n]$ as the following coequalizer

$$\bigsqcup_{0 \le i < j \le n} \Delta[n-2] \rightrightarrows \bigsqcup_{0 \le i \le n} \Delta[n-1] \to \partial\Delta[n]$$

where the top map sends the copy of $\Delta[n-2]$ indexed by (i,j) to the i-th face of the j-th copy of $\Delta[n-1]$ and the bottom map sends the copy of $\Delta[n-2]$ indexed by (i,j) to the $j-1$-th face of the i-th copy of $\Delta[n-1]$

For $0 \leq k \leq n$, we define the k-th horn $\Lambda[n,k]$ to be the smallest subsimplicial set of $\Delta[n]$ that contains all the faces of the non-degenerate n-simplex except the k-th one. It can be defined by a similar coequalizer

$$\bigsqcup_{0 \leq i < j \leq n, i \neq k, j \neq k} \Delta[n-2] \rightrightarrows \bigsqcup_{0 \leq i \leq n, i \neq k} \Delta[n-1] \to \partial\Delta[n]$$

Let I be the set of maps

$$\{\partial\Delta[n] \to \Delta[n], n \in \mathbb{N}\}$$

and J be the set of maps

$$\{\Lambda[n,k] \to \Delta[n], n \geq 1, 0 \leq k \leq n\}$$

Theorem 2.21. *There is a model structure on* `sSet` *in which the cofibrations are generated by I and the trivial cofibrations are generated by J.*

Proposition 2.22. *A map is a cofibration if and only if it is a monomorphism (injective in each degree).*

Proof. Exercise. □

Definition 2.23. A Kan complex is a fibrant simplicial set in this model structure.

Concretely, a Kan complex is a simplicial set X such that, for any diagram

$$\begin{array}{ccc} \Lambda[n,k] & \longrightarrow & X \\ \downarrow & & \\ \Delta[n] & & \end{array}$$

there is a diagonal map $\Delta[n] \to X$ making the tringle commute.

Proposition 2.24. *Let X be a topological space, then $S_\bullet(X)$ is a Kan complex.*

Proof. By adjunction, it suffices to prove that for any diagram

$$\begin{array}{ccc} |\Lambda[n,k]| & \longrightarrow & X \\ \downarrow & & \\ |\Delta[n]| & & \end{array}$$

there exists a digonal map making the diagram commute. This simply comes from the fact that the inclusion $|\Lambda[n,k]| \subset |\Delta[n]|$ has a retraction. □

In order to prove the existence of the model structure, it "suffices" to prove that the weak equivalences (i.e. the maps that can be written as the composite of a trivial cofibration and a trivial fibration) satify the two-out-of-six property. Let us give a more manageable characterization of the weak equivalences.

Proposition 2.25. *Let X be a Kan complex. Consider the factorization*

$$X \cong \mathrm{map}(\Delta[0], X) \to \mathrm{map}(\Delta[1], X) \to \mathrm{map}(\Delta[0] \sqcup \Delta[0], X) \cong X \times X$$

in which the first map is induced by the projection $\Delta[1] \to \Delta[0]$ and the second map is induced by the two face maps. Then this factorization is a path object for X.

Proof. Let us prove that the second map is a fibration. For this, we have to produce a lift in any diagram of the following form

$$\begin{array}{ccc} \Lambda[n,k] & \longrightarrow & \mathrm{map}(\Delta[1],X) \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & \mathrm{map}(\Delta[0]\sqcup\Delta[0],X) \end{array}$$

By adjunction, this amounts to proving that $X \to *$ has the right lifitng property against the map

$$(\Delta[n] \times (\Delta[0] \sqcup \Delta[0])) \sqcup_{\Lambda[n,k]\times(\Delta[0]\sqcup\Delta[0])} (\Lambda[n,k] \times \Delta[1]) \to \Delta[n] \times \Delta[1]$$

This follows from the fact that this map is a relative J-cell complex (this can be proved explicitly but requires some work).

Now, we claim that the first map is a weak equivalence. By the two out of three property, we can instead prove that the map

$$\mathrm{map}(\Delta[1],X) \to \mathrm{map}(\Delta[0],X)$$

given by evaluating at a vertex is a weak equivalence. We shall prove that it is in fact a trivial fibration. As in the first paragraph, we can reduce this to proving that $X \to *$ has the right lifting property against the maps

$$\Delta[n] \sqcup_{\partial\Delta[n]} \partial\Delta[n] \times \Delta[1] \to \Delta[n] \times \Delta[1]$$

This is true since this map is a trivial cofibration (again it can be written as a relative J-cell complex). □

Construction 2.26. Let X be a Kan complex, and A any simplicial set. We denote by $[A,X]$ the set of class of maps $A \to X$ with respect to the right homotopy relation. Explicitly, two maps $f, g : A \to X$ are equivalent if there exists a map

$$h : A \to \mathrm{map}(\Delta[1],X)$$

such that we recover f and g if we compose h with the two evaluation maps

$$\mathrm{map}(\Delta[1],X) \to X \times X$$

Proposition 2.27. *A map $f : A \to B$ in* `sSet` *is a weak equivalence if and only if, for any Kan complex K, the induced map*

$$[B,K] \to [A,K]$$

is a bijection.

Proof. Indeed, by the general theory of model categories, the set $[A,K]$ and $[B,K]$ are the hom sets in the homotopy category. Thus it follows from Yoneda's lemma that the map f is a weak equivalence if and only if the map f induces an isomorphism between the functors represented by A and B. That is for any X, the map

$$\mathrm{Hom}_{\mathtt{Ho(sSet)}}(B,X) \to \mathrm{Hom}_{\mathtt{Ho(sSet)}}(A,X)$$

is a bijection. Since any simplicial set is weakly equivalent to a Kan complex, this happens if and only if it happens for X a Kan complex. □

There is a simpler characterization in terms of simplicial homotopy groups.

Construction 2.28. Consider a Kan complex X and $x \in x_0$. We define the simplicial n-sphere S^n to be the quotient $\Delta[n]/\partial\Delta[n]$. The data of a map $S^n \to X$ is the data of an n-simplex of X whose faces of dimension j with $k \leq n-1$ are all sent to the 0-simplex x (by this we mean the simplex of dimension k induced from x by the unique map $[k] \to [0]$. A pointed right homotopy between two such maps $f, g : S^n \to X$ is a map

$$h : \Delta[n] \to \mathrm{map}(\Delta[1],X)$$

such that

(1) The composition with the two evaluation maps

$$\Delta[n] \to \mathrm{map}(\Delta[1], X) \to X \times X$$

coincides with (f, g).

(2) The restriction of h to $\partial\Delta[n]$ is the constant map with value x.

We denote by $\pi_n(X, x)$ the resulting quotient set.

THEOREM 2.29. *Let $f : X \to Y$ be a map between kan complexes, then, it is a weak equivalence if and only if, for any 0-simplex $x \in X$, the induced map*

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for any n.

3.7. Long exact sequence of homotopy groups. We have the following simplicial analogue of the classical theorem for homotopy groups in a fiber sequence.

THEOREM 2.30. *Let $p : E \to B$ be a fibration between Kan complexes. Let $x \in E_0$ and let $F = \{p(x)\} \times_B E$ be the fiber over $p(x)$. Then we have a long exact sequence of homotopy groups*

$$\ldots \to \pi_n(B, p(x)) \to \pi_{n-1}(F, x) \to \pi_{n-1}(E, x) \to \ldots \to \pi_1(E, x) \to \pi_1(B, p(x))$$

moreover, the sequence can be extended to the right as follows

(1) *There is an action of $\pi_1(B, p(x))$ on the set $\pi_0(F)$ and the image of the map $\pi_1(E, x) \to \pi_1(B, p(x))$ is the stabilizer of $[x] \in \pi_0(F)$.*

(2) *The kernel of the map $\pi_0(F, x) \to \pi_0(E, x)$ (i.e. the inverse image of the class of x) is exactly the orbit of $[x]$ in $\pi_0(F, x)$.*

(3) *The kernel of the map $\pi_0(E, x) \to \pi_0(B, p(x))$ is exactly the image of $\pi_0(F, x) \to \pi_0(E, x)$.*

4. Topological spaces

4.1. Compactly generated weakly Hausdorff spaces. For technical reasons, we shall restrict our class of topological spaces.

DEFINITION 2.31. A k-space is a topological space X whose open subspaces are exactly the subsets $U \subset X$ such that for any compact Hausdorff space C and any map $f : C \to X$, the set $f^{-1}(U)$ is open.

Clearly, we can change the topology of any space to make it satisfy the above definition. This defines a right adjoint to the inclusion

$$\mathtt{kTop} \to \mathtt{Top}$$

of k-spaces into spaces. This functor is often called k-ification. This makes the category of k-spaces into a coreflective subcategory of `Top`.

DEFINITION 2.32. Let `C` be a category. A full subcategory `D` is called a coreflective subcategory if the inclusion

$$\mathtt{D} \to \mathtt{C}$$

is a left adjoint.

In particular, it follows from standard category theory that the category of k-spaces has all limits and colimits. The colimits are simply computed in `Top` while the limits are obtained by applying the right adjoint to the limit computed in `Top`.

We can go one step further with the following definition.

DEFINITION 2.33. The category of compactly generated weakly Hausdorff spaces denoted `CGTop` is the full subcategory of the category of k-spaces on spanned by spaces X that are weakly Hausdorff. That is, for any compact Hausdorff space K and any continuous map $f : K \to X$ the image $f(K)$ is closed in X.

The category of compactly generated weakly Hausdorff spaces is a reflective subcategory of k-spaces (the inclusion is a right adjoint). It follows that it also has all limits and colimits but neither is preserved in general by the inclusion functor $\mathtt{CGTop} \to \mathtt{Top}$. Nevertheless, it can be shown that the colimits that occur in the definition of a relative cell complex are preserved by the inclusion (cf. May's *A concise course in Algebraic Topology*).

The key property that makes CGTop a convenient replacement of Top is the following.

Proposition 2.34. *Let X and Y be two spaces in* CGTop*. We denote by* $\mathrm{map}(X, Y)$ *the set of continuous maps with topology given by the k-ification of the compact-open topology. This makes the category* CGTop *into a cartesian closed category. That is, we have a natural bijection*

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{map}(Y, Z))$$

Proposition 2.35. *The geometric realization of a simplicial set is in* CGTop*.*

Proof. The realization is by definition a colimit. The spaces in this colimit are disjoint unions of simplices which are in CGTop. It remains to show that the colimit is already in CGTop. We leave the verification to the reader. □

Another important reason to use this category is the following proposition.

Proposition 2.36. *Let X and Y be two simplicial sets. Then, the canonical map*

$$|X \times Y| \to |X| \times |Y|$$

is an isomorphism in CGTop*.*

Proof. Since both sides of the equation preserve colimits in Y (because both the category of simplcial sets and the category CGTop are cartesian closed), we may restrict to $Y = \Delta[n]$. Similarly, we may restrict further to $X = \Delta[m]$. So we are reduced to proving the claim for the product of two simplices which can be done explicitly. □

4.2. Internal Hom. From now on, we shall simply write Top for CGTop. We make the following observation on the relationship between internal Homs in sSet and Top.

Proposition 2.37. *Let A be a simplicial set and X be a topological space, then, there is an isomorphism*

$$S_\bullet \,\mathrm{map}(|A|, X) \cong \mathrm{map}(A, S_\bullet(X))$$

Proof. Let U be another simplicial set. We have the following sequence of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}(U, S_\bullet \,\mathrm{map}(|A|, X)) &\cong \mathrm{Hom}(|U|, \mathrm{map}(|A|, X)) \\
&\cong \mathrm{Hom}(|U| \times |A|, X) \\
&\cong \mathrm{Hom}(|U \times A|, X) \\
&\cong \mathrm{Hom}(U \times A, S_\bullet(X)) \\
&\cong \mathrm{Hom}(U, \mathrm{map}(A, S_\bullet(X)))
\end{aligned}$$

This proves the desired result by Yoneda's lemma. □

5. Comparison between Top and sSet

Theorem 2.38. *The Quillen adjunction*

$$|-| : \mathtt{sSet} \leftrightarrows \mathtt{Top} : S_\bullet$$

is a Quillen equivalence.

We shall not give a complete proof but explain the ideas that go into this proof.

(1) First, we show that weak equivalences between Kan complexes are the maps that induce isomorphisms between simplicial homotopy groups.

(2) Then, we observe that for any point in a topological space X, there is a bijection

$$\pi_n(S_\bullet(X), x) \cong \pi_n(X, x)$$

This is not hard to see playing with the adjunction. This implies in particular that the weak equivalences in `Top` are exactly the weak equivalences in the traditional sense (i.e. the maps inducing isomorphisms ofn homotopy groups).

(3) We prove that for any pointed Kan complex (X, x), there is a bijection

$$\pi_n(X, x) \cong \pi_n(|X|, x)$$

constructed as the composite

$$\pi_n(X, x) \to \pi_n(S_\bullet|X|, x) \cong \pi_n(|X|, x)$$

where the first map is induced by the unit of the adjunction and the second map is the bijection of the previous paragraph.

(4) From this, we prove without much trouble that the unit and the counit of the adjuntion are weak equivalences.

Let us give a few more details about part (3). The proof is inductive. It is not too hard to show that this is true for π_0. Then the idea is to construct the simplicial loop space of X. We define first the path space of X by the following pullback square

$$\begin{array}{ccc} P_xX & \longrightarrow & \mathrm{map}(\Delta[1], X) \\ \downarrow & & \downarrow{\scriptstyle (ev_0, ev_1)} \\ X & \xrightarrow[x \times \mathrm{id}_X]{} & X \times X \end{array}$$

Then the map $P_xX \to X$ is a fibration (pullback of a fibration). Moreover, we can alternatively define P_xX by the pullback square

$$\begin{array}{ccc} P_xX & \longrightarrow & \mathrm{map}(\Delta[1], X) \\ \downarrow & & \downarrow{\scriptstyle ev_0} \\ * & \xrightarrow[x]{} & X \end{array}$$

which shows that the map $P_xX \to *$ is an acyclic fibration. In particular P_xX is a Kan complex and has trivial homotopy groups. Let us denote by Ω_xX by the following pullback square

$$\begin{array}{ccc} \Omega_xX & \longrightarrow & P_xX \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ * & \xrightarrow[x]{} & X \end{array}$$

then using the long exact sequence for the fibration $P_xX \to X$, we see that $\pi_n(\Omega_xX, x) \cong \pi_{n+1}(X, x)$. Similarly, one can show (with some work) that the geometric realization of a Kan fibration is a Serre fibration. This implies that there is a compatible isomorphism $\pi_n(\Omega_x|X|, x) \cong \pi_n(|X|, x)$ where $\Omega_x|X|$ denotes the topological loop space of $|X|$.

6. The adjunction between simplicial sets and chain complexes

From a simplicial set, there are two essentially equvialent ways to produce a chain complex. The first way denoted

$$X \mapsto C_*(X, R)$$

is done in two steps.

(1) First apply degreewise the free R-module functor to construct a simplicial R-module.

(2) Then make it into a chain complex by taking as differential the alternating sum of the face maps.

The second one is denoted

$$X \mapsto N_*(X;R)$$

is done as follows

(1) First apply degreewise the free R-module functor to construct a simplicial R-module.
(2) Then take degreewise the quotient by the submodule of degenerate simplices.
(3) Finally give this sequence of R-module the differential given by the alternating sum of face maps.

It turns out that the natural map $C_*(X;R) \to N_*(X;R)$ is a quasi-isomorphism (Exercise). In fact, for $A_\bullet$ a simplicial R-module, we shall denote more generally $N_*(A)$ the chain complex obtained by moding out by the image of all the degeneracies and taking as differential the alternating sum of faces. It turns out that the functor

$$N_* : \mathtt{sAb} \to \mathtt{sSet}$$

is an equivalence of categories. This is called the Dold-Kan correspondance. The inverse of N_* is denoted Γ.

Proposition 2.39. *The functor $N_*(-;R) : \mathtt{sSet} \to \mathtt{Ch}_{\geq 0}(R)$ is a left Quillen functor.*

Proof. It suffices to prove that it sends the generating cofibrations to cofibrations and the generating trivial cofibrations to cofibrations (Exercise). □

In particular, from Ken Brown's lemma, we immediately deduce that this functor preserves all weak equivalences (as all simplicial sets are cofibrant). We can also consider the composition

$$\mathtt{Top} \xrightarrow{S_\bullet} \mathtt{sSet} \xrightarrow{N_*} \mathtt{Ch}_{\geq 0}(R)$$

This is the composition of a right adjoint with a left adjoint. But both functor preserve weak equivalences, it follows that they induce functors at the level of homotopy categories

$$\mathtt{HoTop} \xrightarrow{S_\bullet} \mathtt{HosSet} \xrightarrow{N_*} \mathtt{HoCh}_{\geq 0}(R)$$

where the first functor is an equivalence. So the singular chain functor

$$N_*(-;R) : \mathtt{HoTop} \to \mathtt{HoCh}_{\geq 0}(R)$$

is a left adjoint. Its right adjoint denoted $C_* \mapsto K(C_*)$ can be constructed as follows

(1) Start from an object C_* of $\mathtt{Ch}_{\geq 0}(R)$ and apply the inverse of the Dold-Kan correspondance to make it into a simplicial R-module $\Gamma(C_*)$.
(2) Forget the R-module structure and simply view this object as a simplicial set.
(3) Take geometric realization if one wishes to land in $\mathtt{HoTop}$.

There is a variant of all of this for pointed topological spaces or simplicial set. For (X,x) a pointed simplicial set, we write

$$\tilde{N}_*(X;R) = N_*(X;R)/N_*(x,R)$$

Observe that there is a natural splitting $N_*(X;R) = \tilde{N}_*(X;R) \oplus R$. This means that we can alternatively define $N_*(- : R)$ as the composite

$$\mathtt{sSet} \to \mathtt{sSet}_* \xrightarrow{\tilde{N}_*(-;R)} \mathtt{Ch}_{\geq 0}(R)$$

where the first functor is $X \mapsto X \sqcup *$ (the left adjoint to the forgetful functor $\mathtt{sSet}_* \to \mathtt{sSet}$). We can check that $\tilde{N}_*(-;R)$ is a left Quillen functor exactly as before so we have an adjunction

$$\tilde{N}_*(-;R) : \mathtt{HosSet}_* \leftrightarrows \mathtt{HoCh}_{\geq 0}(R) : K$$

where K is the functor constructed above (observe that the underlying simplicial set of a simplicial R-module is naturally pointed by zero).

A very pleasant consequece of all that have been done so far is the following Theorem.

THEOREM 2.40. *The functor*

$$X \mapsto H^n(X;R)$$

from $\mathtt{HosSet}^{\mathrm{op}}$ *to* $\mathtt{Mod}_R$ *is represented by the space* $K(S(n))$ *(where* $S(n)$ *is the chain complex which is* R *in degree* n *and zero in every other degree).*

PROOF. We have

$$\begin{aligned}\mathrm{Hom}_{\mathtt{HosSet}}(X, K(S(n)) &\cong \mathrm{Hom}_{\mathtt{HoCh}_{\geq 0}(R)}(N_*(X;R), S(n))\\ &\cong H^n(X;R)\end{aligned}$$

Indeed, maps $N_*(X;R) \to S(n)$ can be immediately identified with n-cocycles in $N^*(X;R)$ and the homotopy relation has the effect of quotienting by coboundaries (this is very similar to Corollary 2.8). □

The space $K(S(n))$ is what is called an Eilenberg-MacLane space, it has just one non-trivial homotopy groups in degree n which is isomorphic to R by the following Proposition. More generally, space of the form $K(C)$ are called generalized Eilenberg-MacLane spaces. Their homotopy groups can be understood by the following Proposition.

PROPOSITION 2.41. *For* C *a chain complex, we have a natural isomorphism*

$$\pi_n(K(C)) \cong H_n(C)$$

PROOF. By Corollary 2.8, we have

$$H_n(C) \cong [S(n), C]$$

On the other hand, we have

$$\pi_n(K(C)) = \mathrm{Hom}_{\mathtt{HosSet}_*}(\Delta[n]/\partial\Delta[n], K(C)) \cong [\tilde{N}_*(\Delta[n]/\partial\Delta[n];R), K(C)]$$

So the proof is concluded by observing (Exercise) that there is an isomorphism

$$\tilde{N}_*(\Delta[n]/\partial\Delta[n];R) \cong S(n)$$

□

CHAPTER 3

Homotopy limits and colimits

1. The problem of homotopy colimits and limits

let $\mathtt{M}$ be a category equipped with a notion of weak equivalences. Let I be a small category. There is a diagonal functor $\delta : \mathtt{M} \to \mathtt{M}^I$ which sends an object of $\mathtt{M}$ to the constant diagram in $\mathtt{M}$ on that object. This functor is obviously homotopical (when we say that a map in $\mathtt{M}^I$ is a weak equivalence if it is objectwise a weak equivalence) and so it induces a functor

$$\delta : \mathtt{HoM} \to \mathtt{Ho}(\mathtt{M}^I)$$

Definition 3.1. We say that $\mathtt{M}$ admits I-indexed homotopy colimits if the functor

$$\delta : \mathtt{HoM} \to \mathtt{Ho}(\mathtt{M}^I)$$

admits a left adjoint. In that case, we denote this left adjoint $\mathrm{hocolim}_I$ and call it the homotopy colimit functor.

Dually, we say that $\mathtt{M}$ admits I-indexed homotopy limits if the functor

$$\delta : \mathtt{HoM} \to \mathtt{Ho}(\mathtt{M}^I)$$

admits a right adjoint. In that case, we denote this right adjoint holim_I and call it the homotopy limit functor.

Remark 3.2. The crucial point to understand is that in general, the canonical map $\mathtt{Ho}(\mathtt{M}^I) \to \mathtt{HoM}^I$ is not an equivalence. If it were the case, homotopy limits and colimits would simply be limits and colimits in the homotopy category.

2. The projective and injective model structure

Let D be a small category and $\mathtt{M}$ be a model category.

Definition 3.3. We say that a map in $\mathtt{M}^D$ is a

(1) weak equivalence if it is objectwise a weak equivalence.
(2) projective fibration if it is objectwise a fibration.
(3) injective cofibration if it is objectwise a cofibration.

Definition 3.4. If $\mathtt{M}^D$ with its weak equivalences admits a model structure in which the fibrations are the projective fibrations, we call it the projective model structure.

If $\mathtt{M}^D$ with its weak equivalences admits a model structure in which the cofibrations are the injective cofibrations, we call it the injective model structure.

Without extra assumptions, these model structures might not exist. However, if they exist, they are uniquely determined. The projective cofibrations being defined by the left lifting property against the projective trivial fibrations and the injective fibrations being defined by the right lifting property against the injective trivial cofibrations.

Proposition 3.5. *Let $\mathtt{M}$ be a model structure. Let D be a small category.*

(1) *If the projective model structure exists on $\mathtt{M}^D$, then $\mathtt{M}$ admits D-indexed homotopy colimits.*
(2) *If the injective model structure exists on $\mathtt{M}^D$, then $\mathtt{M}$ admits D-indexed homotopy limits.*

PROOF. Indeed, let us give $\mathtt{M}^D$ the projective model structure. Then the diagonal functor $\delta : \mathtt{M} \to \mathtt{M}^D$ is a right Quillen functor so its left adjoint, the colimit functor, is left derivable by Proposition 1.49 and the left derived functor induces an adjunction

$$\mathbb{L}\operatorname{colim}_D : \mathtt{Ho}(\mathtt{M}^D) \leftrightarrows \mathtt{HoM} : \delta$$

by Proposition 1.50. □

REMARK 3.6. We see from this proposition that, if the projective model structure exists, then the homotopy colimits exist and the homotopy colimit fucntor can be taken to be the left derived functor of the homotopy colimit functor.

Here is a theorem that insures that the projective model structure exists.

THEOREM 3.7. *Let* $\mathtt{M}$ *be a cofibrantly generated model structure. Let* I *and* J *be a set of generating cofibrations and generating trivial cofibrations. Let* D *be a small category. Then the projective model structure on* $\mathtt{M}^D$ *exists and is cofibrantly generated with generating cofibrations the set*

$$I_D = \{\mathrm{id} \otimes f : \mathrm{Hom}_D(d, -) \otimes K \to \mathrm{Hom}_D(d, -) \otimes L, d \in D, f : K \to L \in I\}$$

and generating trivial cofibrations the set

$$J_D = \{\mathrm{id} \otimes f : \mathrm{Hom}_D(d, -) \otimes K \to \mathrm{Hom}_D(d, -) \otimes L, d \in D, f : K \to L \in J\}$$

PROOF. This is an application of Theorem 2.11. There is a right adjoint functor

$$G : \mathtt{M}^D \to \prod_{d \in D} \mathtt{M}$$

sending a functor F to its value at each object. This functor preserves any colimits.

It is easy to show that $\prod_{d \in D} \mathtt{M}$ is cofibrantly generated. For d an object of D and A an object of $\mathtt{M}$, we denote by $M(d)$ the object of $\prod_{d \in D} \mathtt{M}$ which is $\varnothing$ at every factor except at d where it is it M. Then the generating cofibrations of $\prod_{d \in D} \mathtt{M}$ are the maps

$$\{f(d) : K(d) \to L(d), f : K \to L \in I\}$$

and similarly for the generating trivial cofibrations.

If we apply the left adjoint of G to these maps, we obtain the set of maps I_D and J_D.

In order to apply Theorem 2.11, it suffices to check that relative J_D-cell complexes are weak equivalences. But since pushouts are computed objectwise in $\mathtt{M}^D$, this is a straightforward verification. □

3. Homotopy products and coproducts

For S a set, the product model structure on $\mathtt{M}^S$ is exactly the projective and injective model structure. It follows that in model categories, homotopy products and coproducts alwyas exists. Moreover, we have the following proposition.

PROPOSITION 3.8. *Let* $\mathtt{M}$ *be a model category, let* S *be a set. Then the canonical map*

$$\mathtt{Ho}(\mathtt{M}^S) \to (\mathtt{HoM})^S$$

is an equivalence of categories

PROOF. By the general theory of model categories developed in the first chapter, the category $\mathtt{HoM}$ is given by $\pi\mathtt{M}_{cf}$. Similarly the category $\mathtt{Ho}(\mathtt{M}^S)$ is given by $\pi(\mathtt{M}^S)_{cf}$. But it obviously follows from the definition that there is an identification

$$\pi(\mathtt{M}^S)_{cf} \simeq (\pi\mathtt{M}_{cf})^S$$

□

COROLLARY 3.9. *The homotopy product* $\mathtt{Ho}(\mathtt{M}^S) \to \mathtt{HoM}$ *coincides with the product in* $\mathtt{HoM}$ *through the identification of the previous Proposition. The analogous statement holds for the homotopy coproduct.*

PROOF. Indeed the homotopy product is the right adjoint to the diagonal

$$\mathtt{HoM} \to \mathtt{HoM}^S$$

□

In general, in order to compute the homotopy product of a collection $(M_s)_{s \in S}$ in $\mathtt{M}$, one has to first take a fibrant replacement of each object and then take the product (this is the right derived functor of the product). Similarly, in order to compute the homotopy coproduct of a collection $(M_s)_{s \in S}$ in $\mathtt{M}$, one first take a cofibrant replacement of each object and then take the coproduct. In many cases, products or coproducts preserve all weak equivalences and there is no need to derive. For example, coproducts preserve weak equivalences in $\mathtt{Top}$, $\mathtt{sSet}$ and $\mathtt{Ch}_{\geq 0}(R)$. Finite products preserve weak equivalences in these three model categories as well. In the case of $\mathtt{Top}$, infinite products preserve weak equivalences as well.

4. Directed category

Even when the projective model structure exists it is quite unexplicit. In particular, it is hard to say what a cofibrant object is in the projective model structure of $\mathtt{M}^D$, so it is hard to express the homotopy colimit in practice. In this section, we are going to see that under additional assumptions on the indexing category D, we can get a better grasp on the projective model structure on $\mathtt{M}^D$. Moreover, we will also be able to construct the injective model structure.

We denote by $\mathbb{N}$ the totally ordered set of nonnegative integers.

DEFINITION 3.10. A directed category is a small category D with a functor $\alpha : D \to \mathbb{N}$ such that a map in D is sent to an identity map by α if and only if it is an identity map.

REMARK 3.11. We think of α as a “degree function”. The non-identity maps of D are required to strictly increase the degree.

EXAMPLE 3.12. The following examples are directed category.

(1) The categories $[n]$ with α the inclusion.
(2) The category $\mathbb{N}$ with α the identity map.
(3) The category Δ_{inj} of finite totally ordered sets and increasing injections. The degree function is then simply the cardinality function.
(4) The category defining pushouts ; $\{0\} \leftarrow \varnothing \to \{1\}$ with degree function α given by $\alpha(0) = 0$, $\alpha(a) = \alpha(b) = 1$.
(5) More generally the category $P_0(S)$ of strict subsets of a finite set S with morphisms given by inclusions and degree function given by cardinality. Note that the previous example corresponds to the case of a set with 2 elements.

DEFINITION 3.13. Let $F : D \to \mathtt{M}$ be a functor from a directed category to a cocomplete category. Let d be an object of D. The latching object of F at d denoted $L_d F$ is defined by

$$L_d(F) = \operatorname{colim}_{f:c \to d, \alpha(c) < d} F(c)$$

By definition, the latching object comes with a map $L_d(F) \to F(d)$.

THEOREM 3.14. *Let D be a directed category. Then the projective model structure exists on $\mathtt{M}^D$ for any model category $\mathtt{M}$. Moreover the cofibrations are the maps $f : F \to G$ such that for any object d of D, the induced map*

$$F(d) \sqcup_{L_d(F)} L_d(G) \to G(d)$$

are cofibrations.

PROOF. Bicompleteness of $\mathtt{M}^D$ is classical. The weak equivalences satisfy the two-out-of-six property. We shall only prove that the cofibrations and trivial fibrations form a weak factorization system (the case of trivial cofibrations and fibrations is similar). Let $f : F \to G$ be a cofibration and $p : H \to K$ be a trivial fibration. Let

$$\begin{array}{ccc} F & \xrightarrow{u} & H \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow[v]{} & K \end{array}$$

be a commutative square. We shall produce a lift inductively. Our induction hypothesis is as follows : there exists a lift $l : G \to H$ defined on the restrictions of all these functors on $D_{\leq n}$ (the full subcategory of D spanned by object of degree $\leq n$). Clearly this is satisfied at step 0. Indeed the latching object of a functor at an object of degree zero is $\varnothing$. Moreover, there are no non-identity maps between objects of degree zero. So we just have to produce a lift for this diagram evaluating at each object of degree zero which follows from the fact that $\mathtt{M}$ is a model category. Now assume that a lift has been produced up to degree $n-1$ and let d be an object of degree n. The maps u and v and the data of the lift that we already have allow us to construct a commutative square

$$\begin{array}{ccc} F(d) \sqcup_{L_d(F)} L_d(G) & \xrightarrow{u} & H(d) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ G(d) & \xrightarrow[v]{} & K(d) \end{array}$$

for which a lift exists since the left map is a cofibration and the right one is a trivial fibration. All these lifts for all objects of degree n are compatible since there are no non-identity maps between objects of degree n and they give us a lift up to degree n thus completing our proof.

The construction of factorization of $f : F \to G$ is also done inductively. In degree zero this is straightforward. Assume that we have a factorization of the restriction of f to $D_{\leq n-1}$

$$f : F \xrightarrow{i} G' \xrightarrow{p} G$$

Let d be of degree n. Let us consider the map induced by p

$$F(d) \sqcup_{L_d(F)} L_d(G') \to G(d)$$

we can factor this map as a cofibration followed by a trivial fibration

$$F(d) \sqcup_{L_d(F)} L_d(G') \to G'(d) \to G(d)$$

which gives us a definition of $G'(d)$. Clearly these value at all objects d of degree n are compatible so we have a factorization

$$f : F \xrightarrow{i} G' \xrightarrow{p} G$$

defined up to degree n. Moreover, by construction the map $F \to G'$ is a cofibration and the map $G' \to G$ is a trivial fibration. □

Of course all of this story admits a dual version. For a functor $F : D^{\mathrm{op}} \to \mathtt{M}$, the matching object of F at the object d is defined as

$$M_d(F) = \lim_{f:d\to c, c<\alpha(d)} F(c)$$

THEOREM 3.15. *Let D be a directed category. Then the injective model structure exists on $\mathtt{M}^{D^{\mathrm{op}}}$ for any model category $\mathtt{M}$. Moreover the fibrations are the maps $f : F \to G$ such that for any object d of D, the induced map*

$$F(d) \to G(d) \times_{M_d(G)} M_d(F)$$

is a fibration.

5. Homotopy pushouts and pullbacks

5.1. The general theory. We can specialize the work of the previous section to the category $D = P_0\{0,1\} = \{0\} \leftarrow \varnothing \rightarrow \{1\}$.

THEOREM 3.16. *Let* $\mathtt{M}$ *be a model category, then the projective model structure exists on* $\mathtt{M}^D$. *In this model structure, a map*

$$\begin{array}{ccccc} M_0 & \longleftarrow & M_\varnothing & \longrightarrow & M_1 \\ \downarrow & & \downarrow & & \downarrow \\ N_0 & \longleftarrow & N_\varnothing & \longrightarrow & N_1 \end{array}$$

is a cofibration if the following conditions are satisfied

1. $M_\varnothing \to N_\varnothing$ *is a cofibration.*
2. *The map* $N_\varnothing \sqcup_{M_\varnothing} M_0 \to N_0$ *is a cofibration.*
3. *The map* $N_\varnothing \sqcup_{M_\varnothing} M_1 \to N_1$ *is a cofibration.*

In particular, a diagram $M_0 \leftarrow M_\varnothing \rightarrow M_1$ *is cofibrant in this model structure if* $M_\varnothing$ *is cofibrant and the two maps* $M_\varnothing \to M_0$ *and* $M_\varnothing \to M_1$ *are cofibrations.*

THEOREM 3.17. *Let* $\mathtt{M}$ *be a model category, then the injective model structure exists on* $\mathtt{M}^{D^{\mathrm{op}}}$. *In this model structure, a map*

$$\begin{array}{ccccc} M_0 & \longrightarrow & M_\varnothing & \longleftarrow & M_1 \\ \downarrow & & \downarrow & & \downarrow \\ N_0 & \longrightarrow & N_\varnothing & \longleftarrow & N_1 \end{array}$$

is a fibration if the following conditions are satisfied

1. $M_\varnothing \to N_\varnothing$ *is a fibration.*
2. *The map* $M_0 \to M_\varnothing \times_{N_\varnothing} N_0$ *is a fibration.*
3. *The map* $M_1 \to M_\varnothing \times_{N_\varnothing} N_1$ *is a fibration.*

In particular, a diagram $M_0 \rightarrow M_\varnothing \leftarrow M_1$ *is fibrant in this model structure if* $M_\varnothing$ *is fibrant and the two maps* $M_0 \to M_\varnothing$ *and* $M_1 \to M_\varnothing$ *are fibrations.*

These theorems give us a way of computing homotopy pushouts and homotopy pullbacks in model categories. For example, if we want to compute the homotopy pushout of the diagram $M_0 \leftarrow M_\varnothing \rightarrow M_1$ we can apply the following procedure :

1. First (if needed) apply a cofibrant replacement functor so that all objects become cofibrant. Let $M_0' \leftarrow M_\varnothing' \rightarrow M_1'$ be the new diagram.
2. Then factor the maps $M_\varnothing' \to M_0'$ as a cofibration $M_\varnothing' \to M_0''$ followed by a weak equivalence $M_0'' \to M_0'$ and similarly for $M_\varnothing' \to M_1'$.

PROPOSITION 3.18. *Let* $\mathtt{M}$ *be a model category* $f : A \to A'$ *be a map in* $\mathtt{M}$, *then the base change functor*

$$\beta : A/\mathtt{M} \to A'/\mathtt{M}$$

given by $\beta(X) = X \sqcup_A A'$ *is a left Quillen functor. Moreover, it is a left Quillen equivalence if* f *is a trivial cofibration.*

PROOF. The right adjoint to this functor is simply the functor sending $A' \to Y$ to the composite $A \to A' \to Y$. Clearly this functor preserves fibrations and trivial fibrations since those are created by the forgetful functors $A'/\mathtt{M} \to \mathtt{M}$ and $A/\mathtt{M} \to \mathtt{M}$. It follows that β is a left Quillen functor.

Now, assume that f is a trivial cofibration. Let $u : A \to X$ be a cofibrant object of $A/\mathtt{M}$ and $A' \to Y$ be a fibrant object of $A'/\mathtt{M}$. Let $f : X \sqcup_A A' \to Y$ be a weak equivalence. We have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{u} & X & & \\ \downarrow & & \downarrow & \searrow & \\ A' & \longrightarrow & X \sqcup_A A' & \xrightarrow[f]{} & Y \end{array}$$

in which the square is a pushout and the diagonal map is defined so that the triangle commutes. This diagonal map is, by definition, the adjoint to f and it is a weak equivalence as the composite of two weak equivalences (the first one is a trivial cofibration as it is the pushout of a trivial cofibration). Conversely, if the diagonal map is a weak equivalence, the map f must be one as well by the two-out-of-three property. □

Corollary 3.19. *Let* $\mathtt{M}$ *be a model category and consider a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \longrightarrow & X' \end{array}$$

in the square is a pushout square, f *is a weak equivalence between cofibrant objects and* i *is a cofibration, then* g *is a weak equivalence.*

Proof. Tre previous proposition and Ken Brown's lemma imply that the base change functor

$$A/\mathtt{M} \to A'/\mathtt{M}$$

is a Quillen equivalence when $f : A \to A'$ is a weak equivalence between cofibrant objects. The Corollary follows immediately. □

As a corollary, we obtain the following useful proposition.

Proposition 3.20. *Let* $\mathtt{M}$ *be a model category and consider a pushout square in* $\mathtt{M}$

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ A' & \longrightarrow & X' \end{array}$$

with A *and* A' *two cofibrant objects and* i *a cofibration. Then this square is a homotopy pushout square (that is the canonical map from the homotopy pushout to* X' *is a weak equivalence).*

Proof. We observe that X is also cofibrant because of the hypothesis. So using the recipe explained above, the homotopy pushout can be constructed by factoring the map $A \to A'$ as a cofibration $A \to A''$ followed by a weak equivalence $A'' \to A'$ and considering the following pushout square

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & X'' \end{array}$$

these two pushout squares fit into a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ A'' & \longrightarrow & X'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & X' \end{array}$$

in which the bottom square is also a pushout square (this is called the pasting lemma for pushout square and is easy to verify). This bottom square satisfies the hypothesis of the previous Corollary which implies that $X'' \to X'$ is a weak equivalence $\square$

5.2. Homotopy pushout and pullbacks in `sSet` and `Top`. Recall that if $f : A \to B$ is a continuous map, the cylinder of f is given by the following pushout

$$\begin{array}{ccc} A \sqcup A & \longrightarrow & A \times [0,1] \\ {\scriptstyle \mathrm{id}\sqcup f}\downarrow & & \downarrow \\ A \sqcup B & \longrightarrow & \mathrm{Cyl}(f) \end{array}$$

It turns out that, if A is a CW-complex, the top map is a relative CW-complex so in particular a cofibration. It follows that the bottom map is a cofibration as well and in particular, the map $A \to \mathrm{Cyl}(f)$ sending a to $(a, 0)$ is a cofibration. This gives us a factorization of $A \to B$ as

$$A \hookrightarrow \mathrm{Cyl}(f) \xrightarrow{\simeq} B$$

PROPOSITION 3.21. *Let $B \leftarrow A \to C$ be a diagram in* `Top` *with A B and C CW-complexes. Then an explicit model for the homotopy pushout is given by the quotient $B \sqcup A \times [0,1] \sqcup C/\sim$ where the equivalence relation identifies $A \times \{0\}$ with the image of A in B and $A \times \{1\}$ with the image of A in C.*

PROOF. Let us call f the map from $A \to B$ and $g : A \to C$. Let us factor f as

$$A \hookrightarrow \mathrm{Cyl}(f) \xrightarrow{\simeq} B$$

as explained above. Then we can construct the following pushout

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Cyl}(f) \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \longrightarrow & P \end{array}$$

By Proposition 3.20, this is a homotopy pushout square. Moreover the space P is the quotient $B \sqcup A \times [0,1] \sqcup C/\sim$. $\square$

We have the following variant in simplicial sets.

PROPOSITION 3.22. *Let $B \leftarrow A \to C$ be a diagram in* `sSet`*. Then an explicit model for the homotopy pushout is given by the quotient $B \sqcup A \times \Delta[1] \sqcup C/\sim$ where the equivalence relation identifies $A \times \{0\}$ with the image of A in B and $A \times \{1\}$ with the image of A in C.*

An important particular case is given by the suspension functor.

PROPOSITION 3.23. *Let A be a CW-complex. Then the homotopy pushout of $* \leftarrow A \to *$ is the unreduced suspension of A.*

The procedure for homotopy pullbacks is analogous. For $f : A \to B$ a map in `Top`, we define Nf by the following pullback

$$\begin{array}{ccc} Nf & \longrightarrow & \mathrm{map}([0,1], B) \\ \downarrow & & \downarrow \\ A \times B & \xrightarrow[f \times \mathrm{id}]{} & B \times B \end{array}$$

Concretely, a point in Nf is a pair (a, γ) with a a point in A and γ a path in B starting at $f(a)$. Then the projection $Nf \to B$ sending (a, γ) to $\gamma(1)$ is a fibration. This produce a factorization of f as

$$A \xrightarrow{\simeq} Nf \twoheadrightarrow B$$

From this factorization and dualizing the proof above, we deduce the following proposition.

Proposition 3.24. *Let $B \to A \leftarrow C$ be a diagram in* `Top`*. Then an explicit model for the homotopy pullback is given by the subspace of the product $B \times \mathrm{map}([0,1], A) \times C$ of triples (b, γ, c) such that the extremities of γ are the images of b and c respectively.*

Similarly, for `sSet`, we have

Proposition 3.25. *Let $B \to A \leftarrow C$ be a diagram in* `sSet` *in which B and A are Kan complexes. Then an explicit model for the homotopy pullback is given by the sub simplicial set of the product $B \times \mathrm{map}(\Delta[1]], A) \times C$ of triples (b, γ, c) such that the extremities of γ are the images of b and c respectively.*

In pointed spaces, we have the following

Proposition 3.26. *Let $(B, b_0) \to (A, a_0) \leftarrow (C, c_0)$ be a diagram in* $\mathtt{Top}_*$*. Then an explicit model for the homotopy pullback is given by the subspace of the product $B \times \mathrm{map}([0,1], A) \times C$ of triples (b, γ, c) such that the extremities of γ are the images of b and c respectively. The base point is the triple (b_0, a_0, c_0) (where a_0 denotes the constant path at a_0).*

In particular, we have,

Proposition 3.27. *Let (X, x) be a pointed topological space. Then the homotopy pullback of $* \to X \leftarrow *$ is the loop space $\Omega_x X$.*

5.3. Homotopy pushouts and pullbacks in $\mathtt{Ch}_{\geq 0}(R)$.

Proposition 3.28. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a pushout diagram in $\mathtt{Ch}_{\geq 0}(R)$ in which the top map is a monomorphism and the map $A \to A'$ is a quasi-isomorphism. Then the map $B \to B'$ is a quasi-isomorphism.

Proof. The fact that the square is a pushout square implies that the bottom map is also a monomorphism. It also implies that the cokernels of the two horizontal maps are isomorphic. So our square fits in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

in which both rows are exact. The result then follows from the long exact sequence of homology groups and the five lemma. □

From this fact, arguing as in 3.20, we deduce that pushout squares in which only one of the maps is a cofibration are homotopy pushout squares. Wo we need a method for replacing maps by cofibration.

Now, usin gthe cylinder $C \otimes I$ in chain complexes, we have an explicit way of constructing a factorization of a map of chain complexes $f : A \to B$ as a weak equivalence followed by a fibration. It suffices to define the cylinder of f as

$$\mathrm{Cyl}(f) = (A \otimes I) \oplus_A B$$

explicitly, this is the complex given by

$$\mathrm{Cyl}(f)_n = A_n \oplus B_n \oplus A_{n-1}$$

with differential given by $d(a, b, k) = (da - k, db + f(k), -dk)$. There is an obvious inclusion $A \to \mathrm{Cyl}(f)$ sending a to $(a, f(a), 0)$ and a projection $\mathrm{Cyl}(f) \to B$ sending (a, b, k) to $f(a) + b$. If A is cofibrant, this defines a factorization of f as the composite of a cofibration followed by a weak equivalence. Using this factorization, we obtain the following proposition.

Proposition 3.29. *Let $B \xleftarrow{f} A \xrightarrow{g} C$ be a diagram in chain complexes, then an explicit model for the homotopy pushout is given by the complex P with $P_n = B_n \oplus A_{n-1} \oplus C_n$ and differential $d(b,a,c) = (db - fa, -da, dc + ga)$*

Observe that this model for the homotopy pushout contains $B \oplus C$ as a subcomplex. Moreover, the quotient $P/(B \oplus C)$ is identified with $A[-1]$ (the complex A shifted down by 1). We thus have the following Proposition.

Proposition 3.30. *Same notations. There is a long exact sequence of homology groups*

$$\ldots \to H_n(A) \to H_n(B) \oplus H_n(C) \to H_n(P) \to H_{n-1}(A) \to \ldots \to H_0(B) \oplus H_0(C) \to H_0(P) \to 0$$

in which the maps $H_i(A) \to H_i(B) \oplus H_i(C)$ are induced by the map $(f,g) : A \to B \oplus C$.

Proof. This is the long exact sequence for the short exat sequence

$$0 \to B \oplus C \to P \to A[-1] \to 0$$

It suffices to check that the connecting map

$$H_{n-1}(A) \to H_{n-1}(B \oplus C)$$

is induced by the map (f,g). This connecting map applied to a homology class $[a]$ is obtained by picking a lift of a in P_n, taking the differential of that lift and observing that the resulting element is the image of some cycle in $B_{n-1} \oplus C_{n-1}$. A possible lift of a is $(0,a,0)$ whose differential is $(-fa, 0, ga)$ so the connecting map is induced by $(-f,g)$. In order to fix the sign, we can decide to use the map $B \oplus C \to P$ sending (b,c) to $(-b,0,c)$, this does not affect the cokernel and give us the desired long exact sequence. □

6. Preservation of homotopy colimits by Quillen left adjoints

Let $\mathtt{M}$ and $\mathtt{N}$ be two model categories and

$$F : \mathtt{M} \leftrightarrows \mathtt{N} : G$$

be a Quillen adjunction. Let D be a small category. Assume that the projective model structure exists on $\mathtt{M}^D$ and $\mathtt{N}^D$, then, objectwise application of F and G induces a Quillen adjunction

$$F : \mathtt{M}^D \leftrightarrows \mathtt{N}^D : G$$

Theorem 3.31. *There is a natural isomorphism*

$$\mathbb{L}F(\operatorname{hocolim}_D A) \cong \operatorname{hocolim}_D \mathbb{L}F(A)$$

Proof. Before giving the proof, we make the observation that the two symbols $\mathbb{L}F$ mean different things in this equation. On the left, it is the functor $\mathbb{L}F : \mathtt{HoM} \to \mathtt{HoN}$ and on the right, it is the functor $\mathbb{L}F : \mathtt{Ho}(\mathtt{M}^D) \to \mathtt{Ho}(\mathtt{N}^D)$.

We have a square of left Quillen functors

$$\begin{array}{ccc} \mathtt{M}^D & \xrightarrow{F} & \mathtt{N}^D \\ {\scriptstyle \operatorname{colim}_D}\downarrow & & \downarrow{\scriptstyle \operatorname{colim}_D} \\ \mathtt{M} & \xrightarrow[F]{} & \mathtt{N} \end{array}$$

which commutes up to natural isomorphism. So let us take $\tilde{A} \to A$ a cofibrant replacement of A in $\mathtt{M}^D$, we have

$$F(\operatorname{colim}_D \tilde{A}) \cong \operatorname{colim}_D F(\tilde{A})$$

since $\operatorname{colim}_D \tilde{A}$ is cofibrant and $F(\tilde{A})$ is also cofibrant, we have the required isomorphism in the homotopy category. □

As a corollary, we have the following result.

THEOREM 3.32. *Let*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T \end{array}$$

be a homotopy pushout square in $\mathtt{sSet}$ *or* $\mathtt{Top}$. *Then the induced square*

$$\begin{array}{ccc} N_*(X;R) & \longrightarrow & N_*(Y;R) \\ \downarrow & & \downarrow \\ N_*(Z;R) & \longrightarrow & N_*(T;R) \end{array}$$

is a homotopy pushout square in $\mathtt{Ch}_{\geq 0}(R)$. *In particular, we have a long exact sequence of homology groups*

$$\ldots \to H_n(X) \to H_n(Y) \oplus H_n(Z) \to H_n(T) \to H_{n-1}(X) \to \ldots \to H_0(T) \to 0$$

7. Homotopy colimits indexed by Δ^{op}

In this section, we assume that our model category has a simplicial structure. That is there is a pairing

$$\mathtt{sSet} \times \mathtt{M} \to \mathtt{M}$$

denoted $(K, M) \mapsto K \otimes M$ that is a left adjoint in both variables. That is, there is a functor

$$\mathrm{map}_{\mathtt{M}}(-,-) : \mathtt{M}^{\mathrm{op}} \times \mathtt{M} \to \mathtt{sSet}$$

and a natural isomorphism

$$\mathrm{Hom}_{\mathtt{M}}(K \otimes M, N) \cong \mathrm{Hom}_{\mathtt{sSet}}(K, \mathrm{map}(M, N))$$

and a functor

$$\mathtt{M} \times \mathtt{sSet}^{\mathrm{op}} \to \mathtt{M}, (M, K) \mapsto M^K$$

and a natural isomorphism

$$\mathrm{Hom}_{\mathtt{M}}(K \otimes M, N) \cong \mathrm{Hom}_{\mathtt{M}}(M, N^K)$$

We make the final requirement that $* \otimes M$ is naturally isomorphic to M. This is equivalent to asking that the 0-simplices of $\mathrm{map}(M, N)$ are naturally isomorphic to $\mathrm{Hom}(M, N)$ or that M^* is naturally isomorphic to M.

DEFINITION 3.33. A model category $\mathtt{M}$ with a simplicial structure is a simplicial model category if for any cofibration $f : K \to L$ in $\mathtt{sSet}$ and any cofibration $g : M \to N$ in $\mathtt{M}$, the induced map

$$K \otimes N \sqcup_{K \otimes M} L \otimes M \to L \otimes N$$

is a cofibration. Moreover, this cofibration is a trivial cofibration if one of the two maps f and g is one.

REMARK 3.34. If $\mathtt{M}$ is cofibrantly generated (which is the case in the examples below), it is enough to check the definition for $f : K \to L$ and $g : M \to N$ generating cofibrations and then for one of them a generating trivial cofibration.

EXAMPLE 3.35. Here are a few examples.

(1) The category $\mathtt{sSet}$ with the tensor product simply given by the product.
(2) The category $\mathtt{sSet}_*$ with the tensor product simply given by $K \otimes X \cong K_+ \wedge X$ where $K_+ = K \sqcup *$.
(3) The category $\mathtt{Top}$ with $K \otimes X \cong |K| \times X$.
(4) The category $\mathtt{Top}_*$ with $K \otimes X \cong |K_+| \wedge X$.

REMARK 3.36. in this section, we will have to consider simplicial objects in simplicial sets. In order to not confuse the two simplicial directions, we shall write $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$ for this category and we will write the "internal" simplicial direction as a subscript and the "diagramatic" direction as an argument. For instance $X_p[n]$ denotes the p-simplices of the value at $[n]$ of a simplicial object X in $\mathtt{sSet}$. When we want to view a simplicial object in $\mathtt{sSet}$ as a bisimplicial set, we shall write the internal direction first.

DEFINITION 3.37. Let $\mathtt{M}$ be a simplicial model category. Let X be a simplicial object of $\mathtt{M}$. The geometric realization of X denoted $|X|$ is the coend $\Delta[-] \otimes_{\Delta^{\mathrm{op}}} X$.

EXAMPLE 3.38. For a simplicial object in $\mathtt{sSet}$, the geometric realization is the diagonal simplicial set

$$|X|_n = X_n[n]$$

Indeed, we have

$$|X|_n = \Delta[-]_n \otimes_{\Delta^{\mathrm{op}}} X_n = \mathrm{Hom}_{\Delta}([n], -) \otimes_{\Delta^{\mathrm{op}}} X_n(-) = X_n[n]$$

by the co-Yoenda lemma (Proposition 2.13).

EXAMPLE 3.39. If X is a simplicial set views as a simplicial object in discrete topological spaces, its geometric realization is indeed the geometric realization of the simplicial set.

PROPOSITION 3.40. *Let D be a small category and $F : D \to \mathtt{M}$ be a functor to a cocomplete category. Then we have an isomorphism*

$$\mathrm{colim}_D F \cong * \otimes_D F$$

where $*$ denotes the constant functor $D^{\mathrm{op}} \to \mathtt{Set}$ with value $*$.

PROOF. This coend is the following coequalizer

$$\bigsqcup_{f:d \to d'} F(d) \rightrightarrows \bigsqcup_{d} F(d)$$

which is well-known to compute the colimit. □

PROPOSITION 3.41. *Let $\mathtt{M}$ be a simplicial model category. Consider the coend pairing $\mathtt{sSet}^{D^{\mathrm{op}}} \times \mathtt{M}^D \to \mathtt{M}$ given by*

$$(F, G) \mapsto F \otimes_D G$$

then

1. *If we fix any F, this functor preserves weak equivalences between projectively cofibrant objects in $\mathtt{M}^D$.*
2. *If we fix a projectively cofibrant G, this functor preserve all weak equivalence in the F variable.*
3. *If we fix an objectwise cofibrant G, this functor preserves weak equivalences between projectively cofibrant functors F.*
4. *If we fix a projectively cofibrant F, this functor preserves all weak equivalences between objectwise cofibrant functors G.*

PROOF. Part (1) follows from the fact that $F \otimes_D -$ is a left Quillen functor from M^D with the projective model structure to $\mathtt{M}$. For this it suffices to prove that it sends generating (trivial) cofibrations to (trivial) cofibrations which can be checked easiy.

Now, we prove part (2). Let us call good a functor $G : D \to \mathtt{M}$ for which the conclution holds. Clearly good functors are stable under retracts, colimits. They contain the initial object. In order to prove the claim, using the small object argument, we can thus reduce to proving that good functors are stable under gluing I_D-cells. That is, for $f : \mathrm{Hom}_D([d], -) \otimes K \to \mathrm{Hom}_{\Delta^{\mathrm{op}}}([d], -) \otimes L$ a generating cofibration in $\mathtt{M}^D$, if we have a pushout square

$$\begin{array}{ccc} \mathrm{Hom}_{\Delta^{\mathrm{op}}}([d], -) \otimes K & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\Delta^{\mathrm{op}}}([d], -) \otimes L & \longrightarrow & G' \end{array}$$

in which G is good, then G' is also good. Let $F \to F'$ be a weak equivalence between objects of $\mathtt{sSet}^{D^{\mathrm{op}}}$. Appling $F \otimes_D -$ and $F' \otimes_D -$ to this square we get two pushout squares in $\mathtt{M}$

$$\begin{array}{ccc} F(d) \otimes K & \longrightarrow & F \otimes_D G \\ \downarrow & & \downarrow \\ F(d) \otimes L & \longrightarrow & F \otimes_D G' \end{array} \qquad \begin{array}{ccc} F'(d) \otimes K & \longrightarrow & F' \otimes_D G \\ \downarrow & & \downarrow \\ F(d) \otimes L & \longrightarrow & F' \otimes_D G' \end{array}$$

These two pushout squares are actually homotopy pushout squares since the left vertical map is a cofibration in each case and the top two objects are cofibrant (because of part (1)), it follows that the induced map $F \otimes_D G' \to F' \otimes_D G'$ is a weak equivalence as desired.

Finally part (3) is proven analogously to part (1) and part (4) is proven analogously to part (2). □

Corollary 3.42. *Let $F : \Delta^{\mathrm{op}} \to \mathtt{M}$ be a projectively cofibrant simplicial object in $\mathtt{M}$. Then the geometric realization of F is a model for the homotopy colimit of F*

Proof. From Proposition 3.41, we see that if F is projectively cofibrant, the canonical map

$$\Delta[-] \otimes_{\Delta^{\mathrm{op}}} F \to * \otimes_{\Delta^{\mathrm{op}}} F \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} F$$

is a weak equivalence. □

The advantage of the geometric realization is that it will compute the homotopy colimit much more generally than for projectively cofibrant diagrams.

7.1. The Reedy model structure. We denote by Δ_{inj} and Δ_{surj} the full subcategories of injections and surjections in Δ. Observe that Δ_{inj} and $\Delta_{sur}^{\mathrm{op}}$ are directed categories (with degree function the cardinality). For $X \in \mathtt{M}^{\Delta^{\mathrm{op}}}$, we define the latching object of X at $[n]$, denoted $L_n X$, to be the latching object of the restriction of X to $\Delta_{surj}^{\mathrm{op}}$. We define the matching object at $[n]$, denoted $M_n X$ to be the matching object of the restriction of X to $\Delta_{inj}^{\mathrm{op}}$.

Theorem 3.43. *There is a model structure on $\mathtt{M}^{\Delta^{\mathrm{op}}}$ in which*

(1) *The weak equivalences are the objectwise weak equivalences.*
(2) *The cofibrations are the maps $f : F \to G$ whose restriction to $\Delta_{surj}^{\mathrm{op}}$ are projective cofibrations. That is, for all $[n]$, the map*
$$F[n] \sqcup_{L_n F} L_n G \to G[n]$$
is a cofibration.
(3) *The fibrations are the maps whose restriction to $\Delta_{inj}^{\mathrm{op}}$ are injective fibrations. That is for all $[n]$, the map*
$$F[n] \to G[n] \times_{M_n G} M_n F$$
is a fibration.

Remark 3.44. One can show (this is not completely trivial) that a Reedy cofibration is an injective cofibration and that a Reedy fibration is a projective fibration. So the Reedy model structure sits between the injective and projective model structures. It has more cofibrations than the projective one (in particular more cofibrant objects) but fewer fibrations and it has more fibrations that the injective one and fewer cofibrations. Nevertheless, all three categories share the same weak equivalences so they all have the same homotopy category.

Proposition 3.45. *The geometric realization functor $\mathtt{M}^{\Delta^{\mathrm{op}}} \to \mathtt{M}$ is a left Quillen functor from the Reedy model structure.*

Proof. The right adjoint is given by $M \mapsto M^{\Delta[-]}$. It suffices to prove that this right adjoint preserves fibrations and trivial fibrations. For this, we make the observation that the matching object of $M^{\Delta[-]}$ coincides with $M^{\partial\Delta[n]}$.

So now, consider a (trivial) fibration $X \to Y$ in $\mathtt{M}$, the induced map

$$X^{\Delta[n]} \to Y^{\Delta[n]} \times_{X^{\partial\Delta[n]}} Y^{\partial\Delta[n]}$$

is a trivial fibration because of the axiom SM7. □

In particular, we deduce the following consequence.

Proposition 3.46. *Let $\mathtt{M}$ be a simplicial model category. Let $F \in \mathtt{M}^{\Delta^{\mathrm{op}}}$ be a simplicial diagram in $\mathtt{M}$ which is Reedy cofibrant. Then the geometric realization of F is a model for the homotopy colimit of F.*

Proof. Take a projectively cofibrant replacement $\tilde{F} \to F$. We have a zig-zag in $\mathtt{M}$

$$|F| \leftarrow |\tilde{F}| \to * \otimes_{\Delta^{\mathrm{op}}} \tilde{F} \cong \operatorname{hocolim}_{\Delta^{\mathrm{op}}} F$$

where the left pointing arrow is a weak equivalence because of the previous proposition and Ken Brown's lemma. □

Proposition 3.47. *In $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$ all objects are Reedy cofibrant.*

Proof. Let $[n] \mapsto X[n]$ be an object of $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$. We wish to show that the map $L_n(X) \to X[n]$ is a cofibration for all n. Since the cofibrations in $\mathtt{sSet}$ are exactly the degreewise injective maps and since colimits are computed degreewise, we see that it is enough to prove that for all k and n, the map

$$\operatorname{colim}_{[n] \twoheadrightarrow [p], p \neq n} X_k[p] \to X_k[n]$$

is injective (the colimit is taken over surjective maps). In other words, we are reduced to prove that for $Y_\bullet$ a simplicial *set*, the map

$$\operatorname{colim}_{[n] \twoheadrightarrow [p], p \neq n} Y_p \to Y_n$$

is injective. This follows from the so-called Eilenberg-Zilber lemma (a proof can be found on page 26-27 of Gabriel-Zisman *Calculus of fractions and homotopy theory*) which asserts that in a simplicial set, for every n-simplex σ, there exists a unique non-degenerate simplex τ and a surjective map $[n] \twoheadrightarrow [p]$ with $[p]$ the dimension of τ such that σ is the image of τ under this map. □

Corollary 3.48. *Let X be a simplicial set. View X as a functor $\Delta^{\mathrm{op}} \to \mathtt{sSet}$ given by sending $[n]$ to the constant simplicial set X_n. Then we have a natural weak equivalence*

$$\operatorname{hocolim}_{[n] \in \Delta^{\mathrm{op}}} X_n \simeq X$$

Proof. because of the previous Proposition and Proposition 3.46, there is a natural weak equivalence

$$|[n] \mapsto X_n| \simeq \operatorname{hocolim}_{[n]} X_n$$

On the other hand, the geometric realization in $\mathtt{sSet}^{\Delta^{\mathrm{op}}}$ is simply the diagonal so the left-hand side is isomorphic to X. □

8. The fundamental theorem of homotopy theory

Theorem 3.49. *Let $\mathtt{M}$ be a simplicial cofibrantly generated model category, let $F : \mathtt{sSet} \to \mathtt{M}$ be a homotopical functor that preserves homotopy colimits. Then F is naturally weakly equivalent to the functor*

$$X \mapsto X \otimes^{\mathbb{L}} F(*)$$

Proof. Before going into the proof, let us clarify what we mean by preservation of homotopy colimits. Let $\mathtt{M}$ be a cofibrantly generated model category and let D be a small category and suppose that we have chosen a cofibrantly replacement functor in $\mathtt{M}^D$ with its projective model structure written $A \mapsto \tilde{A}$ with a natural weak equivalence $\tilde{A} \to A$. Then we define $\operatorname{hocolim}_D : \mathtt{M}^D \to \mathtt{M}$ to be $A \mapsto \operatorname{colim}_D \tilde{A}$. In particular, there is a natural transformation $\operatorname{hocolim}_D A \to A$. Now, let $F : \mathtt{M} \to \mathtt{N}$ be a weak equivalence preserving functor between two cofibrantly generated model categories. We say that it preserves homotopy colimits indexed by D if the zig-zag of natural transformations

$$\operatorname{hocolim}_d F \circ A \xleftarrow{\simeq} \operatorname{hocolim}_d F \circ \tilde{A} \to \operatorname{colim}_d F \circ \tilde{A} \to F(\operatorname{colim}_d \tilde{A}) := F(\operatorname{hocolim}_d A)$$

is a weak equivalence for all A. One can check that this is independant on the choice of cofibrant replacement made (Exercise).

Now, we prove the theorem. First of all, since the simplicial set X is naturally weakly equivalent to $\operatorname{hocolim}_{[n]} X_n$, we have a natural zig-zag of weak equivalence

$$X \otimes^{\mathbb{L}} F(*) \simeq (\operatorname{hocolim}_{[n]} X_n) \otimes^{\mathbb{L}} F(*) \simeq \operatorname{hocolim}_{[n]}(X_n \otimes^{\mathbb{L}} F(*))$$

where the second weak equivalence comes from the fact that $- \otimes F(*)$ is a left Quillen functor. Since F preserves homotopy coproducts, the natural map $X_n \otimes^{\mathbb{L}} F(*) \to F(X_n)$ is a weak equivalence and so induces a natural weak equivalence

$$\operatorname{hocolim}_{[n]} X_n \otimes F(*) \xrightarrow{\simeq} \operatorname{hocolim}_{[n]} F(X_n)$$

Finally, since F preserves homotopy colimits indexed by Δ^{op}, we have a natural zig-zag of weak equivalence

$$\operatorname{hocolim}_{[n]} F(X_n) \simeq F(\operatorname{hocolim}_{[n]} X_n) \simeq F(X)$$

□

There is an analogous version of this theorem when the target is $\mathtt{Ch}_{\geq 0}(R)$ (which is not a simplicial model category).

Theorem 3.50. *Let $F : \mathtt{sSet} \to \mathtt{Ch}_{\geq 0}(R)$ be a homotopical functor that preserves homotopy colimits. Then F is naturally weakly equivalent to the functor*

$$X \mapsto N_*(X; R) \otimes^{\mathbb{L}}_R F(*)$$

Proof. This is proved exactly as above once we observe that for a constant simplicial set S, there is a natural isomorphism $N_*(S; R) \otimes_R C_* \cong S \otimes C$. □

9. Other homotopy colimits

Construction 3.51. Let D be a small category. Let $F : D^{\mathrm{op}} \to \mathtt{sSet}$ and $G : D \to \mathtt{M}$ be two functors. The two-sided bar construction denote $B(F, D, G)$ is the simplicial object in $\mathtt{M}$ given by

$$[n] \mapsto \coprod_{d_0, d_1, \ldots, d_n} G(d_n) \times \operatorname{Hom}(d_n, d_{n-1}) \times \ldots \times \operatorname{Hom}(d_1, d_0) \otimes F(d_0)$$

where the disjoint union is taken over all sequences of $n+1$ objects in D.

The n-th face map maps the factor indexed by $d_0, \ldots, d_n$ to the factor indexed by $d_0, \ldots, d_{n-1}$ and is given by

$$G(d_n) \times \operatorname{Hom}(d_n, d_{n-1}) \times \ldots \times \operatorname{Hom}(d_1, d_0) \otimes F(d_0) \to$$

$$G(d_{n-1}) \times \operatorname{Hom}(d_{n-1}, d_{n-2}) \times \ldots \times \operatorname{Hom}(d_1, d_0) \otimes F(d_0)$$

where we have used the map $G(d_n) \times \operatorname{Hom}(d_n, d_{n-1}) \to G(d_{n-1})$ induced by the fact that G is a covariant functor. Similarly, the n-th face maps is the map

$$G(d_n) \times \operatorname{Hom}(d_n, d_{n-1}) \times \ldots \times \operatorname{Hom}(d_1, d_0) \otimes F(d_0) \to$$

$$G(d_n) \times \operatorname{Hom}(d_n, d_{n-1}) \times \ldots \times \operatorname{Hom}(d_2, d_1) \otimes F(d_1)$$

is induced by the fact that F is a contravariant functor. And the i-th face for $i \in \{1, \dots, n-1\}$ is simply induced by the composition map

$$\mathrm{Hom}(d_{i+1}, d_i) \times \mathrm{Hom}(d_i, d_{i-1}) \to \mathrm{Hom}(d_{i+1}, d_{i-1})$$

The degeneracy maps are induced by using the identity map $* \to \mathrm{Hom}(d_i, d_i)$ in all possible places.

EXAMPLE 3.52. Suppose that $\mathtt{M} = \mathtt{sSet}$ and that F and G are both the constant functor with value the point, then $B(*, D, *)$ is simply the nerve of the category D.

We make a small abuse of notation and write D for the functor $D^{\mathrm{op}} \to \mathrm{Fun}(D, \mathtt{Set})$ sending d to $\mathrm{Hom}(d, -)$. Then we have the following formula.

PROPOSITION 3.53. *There is an isomorphism of simplicial objects in* $\mathtt{M}$*.*

$$B(F, D, G) \cong B(F, D, D) \otimes_D G$$

There is an isomorphism in $\mathtt{M}$

$$|B(F, D, G)| \cong |B(F, D, D)| \otimes_D G$$

The main theorem that makes this construction useful is the following.

THEOREM 3.54. *For any functor* $F : \mathtt{D}^{\mathrm{op}} \to \mathtt{sSet}$*, the canonical map*

$$|B(F, D, D)| \to F$$

is a cofibrant replacement in the projective model structure on $\mathtt{sSet}^{D^{\mathrm{op}}}$*.*

PROOF. Since geometric realization is a left Quillen functor from the Reedy model structure, we see that it is enough to prove the following two facts

(1) $B(F, D, D)$ is Reedy cofibrant in $(\mathtt{sSet}^{D^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$ (where we use the projective model structure on $\mathtt{sSet}^{D^{\mathrm{op}}}$).
(2) The canonical map $|B(F, D, D)| \to F$ is a weak equivalence.

For fact (1), we can alternatively rewrite

$$B_n(F, D, G) = \bigsqcup_{d_0 \to d_1 \to \dots \to d_n} G(d_n) \otimes F(d_0)$$

where the coproduct is indexed by all n-tuples of composable arrows in D. Then the map from the latching object into the n-simplices is simply the inclusion of the summand indexed by degenerate sequences (i.e. those in which one of the maps is an identity). This inclusion is a cofibration as all objects are cofibrant in $\mathtt{sSet}$.

Fact (2) is a consequence of a more general fact. It turns out that the simplicial object $B(F, D, D)$ can be extended to the larger category Δ_-^{op}. The category Δ_- is the category with objects $[i]$ for all i in $\mathbb{N} \sqcup \{-1\}$ and with morphisms $[i] \to [j]$ the maps of totally ordered sets

$$\{-1, 0, \dots, i\} \to \{-1, 0, \dots, j\}$$

that send -1 to -1 (with the convention that $[-1]$ corresponds to the set $\{-1\}$). We can view Δ as a subcategory of Δ_- by sending $[i]$ to $[i]$ an extending a map $f : [i] \to [j]$ to the unique map that coincides with f on $\{0, \dots, i\}$ and sends -1 to -1.

Then, we claim that $B(F, D, D)$ extends to this larger category (with $[-1]$ being send to F). This property of an augmented simplicial object is called admitting a contracting homotopy. When this is the case, the augmentation map is a homotopy equivalence (see Corollary 4.5.2 in Emily Riehl's *Categorical homotopy theory*). □

COROLLARY 3.55. *Let* $G : D^{\mathrm{op}} \to \mathtt{M}$ *be an objectwise cofibrant functor in* $\mathtt{M}^D$*. Then the homotopy colimit of* F *is modelled by* $|B(*, D, G)|$*.*

PROOF. Indeed, by the previous Theorem, the map

$$|B(*, D, D)| \to *$$

is a cofibrant replacement in the projective model structure. Let $\tilde{G} \to G$ be a cofibrant replacment in the projective model structure on $\mathtt{M}^D$, then we have

$$\mathrm{hocolim}_D\, G := \mathrm{colim}_D\, \tilde{G} = * \otimes_D \tilde{G} \simeq |B(*, D, D)| \otimes_D \tilde{G} \simeq |B(*, D, D)| \otimes_D G \cong |B(*, D, G)$$

where the first equivalence and the second equivalence follow from Proposition 3.41. □

REMARK 3.56. This fact can be viewed as a homotopical analogue of the classical fact that any colimit can be written in terms of coproducts and a reflexive coequalizer. In homotopy theory, the notion of reflective coequalizer is replaced by homotopy colimit of a simplicial diagram. Note in particular, that the truncation in cardinality ≤ 1 of the bar construction $B(*, D, G)$ is the standard reflective coequalizer computing the colimit of G.

10. Homotopy orbits and homotopy fixed points

Let $\mathtt{M}$ be a complete and cocomplete category. Consider a discrete group G and an object X of $\mathtt{M}$ equipped with a G-action. This data can be encoded as a functor $\mathscr{B}G \to \mathtt{M}$ where $\mathscr{B}G$ is the category with one objects and G as set of morphisms of this object. We denote by X_G the orbits (i.e. colimit of the diagram $\mathscr{B}G \to \mathtt{M}$ that classifies the G-action) and by X^G the fixed points (the limit of this diagram). The terminology coinvariants and invariants can also be found in the literature.

DEFINITION 3.57. If $\mathtt{M}$ is a model category, we denote by X_{hG} and X^{hG} the homotopy colimit and limit of the diagram $\mathscr{B}G \to \mathtt{M}$ classifying the G-action. We call X_{hG} the homotopy orbits and X^{hG} the homotopy fixed points.

Of course the homotopy orbits and homotopy fixed points only exists under the standard assumptions seen in the previous sections. In particular, we can specilize the previous section to this case and we deduce the following constructuction.

CONSTRUCTION 3.58. If $\mathtt{M}$ is a simplicial model category, the homotopy orbits can be constructed (if X is cofibrant) as the geometric realization of the bar construction $B(*, G, X)$

$$[n] \mapsto G^n \otimes X$$

with face maps given by using the action of G on X or by multiplying two copies of G together. The degeneracy maps are given by inserting the neutral element of G in all the possible places.

We can understand why this construction works (in $\mathtt{sSet}$) thanks to the following proposition.

PROPOSITION 3.59. *The following fact hold in the model category* $\mathtt{sSet}^G$.

(1) *A G simplicial set whose G-action is free in each simplicial degree is cofibrant in the projective model structure* $\mathtt{sSet}^G$.
(2) *For X a G-simplicial set, the bar construction*

$$B(G, G, X) : [n] \mapsto G^{n+1} \otimes X$$

is such that $|B(G, G, X)| \to X$ is a weak equivalence with free source.
(3) *We have* $|B(G, G, X)|_G \cong |B(*, G, X)|$.

PROOF. The first fact can be checked by proving that the cell complexes in the projective model structure are exactly the free G-simplcial sets.

The second point comes from the identification

$$B(G, G, *) \times X \cong B(G, G, X)$$

This identification is not completely immediate. It is given in simplicial degree n by

$$(g_0, \dots, g_n, x) \mapsto (g_0, \dots, g_n, g_{n-1}^{-1}x)$$

Therefore it is enough to show that the simplicial set $EG = B(G, G, *)$ is contractible (as it obviously has a free G-action). This follows from the fact that this simplicial set is the nerve of the category of translations in G which is equivalent to the category $[0]$ and the following lemma.

Finally the last point is easy as it is already true before taking geometric realization (and geometric realization commutes with colimits). □

LEMMA 3.60. *Let C and D be two small categories, $f : C \to D$ and $g : C \to D$ be two functors and $H : f \to G$ be a natural transformation. Then the two maps Nf and Ng are homotopic. In particular, if $f : C \leftrightarrows D : g$ is an adjunction then the two map Nf and Ng are mutually inverse weak equivalences of simplicial sets.*

PROOF. A natural transformation can be viewed as a functor $H : C \times [1] \to D$ such that $H(c, 0) = f(c)$ and $H(c, 1) = g(c)$. This realizes to a homotopy between Nf and Nc since N preserves products and $N[1] \cong \Delta[1]$. □

The simplicial sets EG and BG play a fundamental role in homotopy theory. The map $EG \to BG$ is a Kan fibration and it is the universal principa G-bundle (i.e. a principal G-bundle over a simplicial set X is obtained by pulling back the universal one. Let us also observe that the long exact sequence for the fiber sequence

$$G \to EG \to BG$$

implies that BG is an Eilenberg-MacLane space of type $K(G, 1)$.

CHAPTER 4

Introduction to spectra

In this chapter, we denote by $\mathtt{S}$ either the category of simplicial sets or the category of topological spaces. We shall call an element of $\mathtt{S}$ a "space".

1. Recollections about pointed spaces

Recall that the category $\mathtt{S}_*$ admits a model structure in which the weak equivalences are the maps that become weak equivalences after forgetting the base point. This category is equipped with a mapping space $\mathrm{map}_*(X,Y)$ given by the subspace of $\mathrm{map}(X,Y)$ of maps preserving the base point. There is also the smash product $(X,Y)\mapsto X\wedge Y$. These two functors form a two variable adjunctions :

$$\mathrm{Hom}_{\mathtt{S}_*}(X\wedge Y, Z)\cong \mathrm{Hom}_{\mathtt{S}_*}(X, \mathrm{map}(Y,Z))$$

We can specialize this adjunction to $X=S^1$ (in $\mathtt{sSet}_*$ we take $\Delta[1]/\Delta[0]\sqcup\Delta[0]$ as a model for S^1). We use the notation $X\mapsto \Sigma X$ for the smash product of X with S^1 and $X\mapsto \Omega X$ for the pointed mapping space from S^1. So we have an adjunction

$$\mathrm{Hom}_{\mathtt{S}_*}(\Sigma X, Y)\cong \mathrm{Hom}_{\mathtt{S}_*}(X, \Omega Y)$$

Proposition 4.1. *Let X be a cofibrant object in $\mathtt{S}_*$, then $Y\mapsto Y\wedge X$ is a left Quillen functor*

Proof. We do the proof in $\mathtt{sSet}$. Clearly, if $A\to B$ is a monomorphism, so is $A\wedge X\to B\wedge X$ so the functor preserves cofibrations. Now, we shall prove that smashing with X preserves all weak equivalences. Let $f:A\to B$ be a weak equivalence. Consider the two squares

$$\begin{array}{ccc} X\vee A & \longrightarrow & X\times A \\ \downarrow & & \downarrow \\ * & \longrightarrow & X\wedge A \end{array} \qquad\qquad \begin{array}{ccc} X\vee B & \longrightarrow & X\times B \\ \downarrow & & \downarrow \\ * & \longrightarrow & X\wedge B \end{array}$$

These two squares are pushout squares and in each case, the top map is a monomorphism, it follows that they are both homotopy pushout squares. Therefore, by homotopy invariance of homotopy pushouts, it suffices to prove that $X\vee A\to X\vee B$ and $X\times A\to X\times B$ are weak equivalences. But it turns out that, in $\mathtt{sSet}$, weak equivalences are stable under products and coproducts. From this, we easily deduce the result. □

Corollary 4.2. *The suspension-loop adjunction is a Quillen adjunction.*

Proposition 4.3. *For X cofibrant, the suspension ΣX is a model for the homotopy pushout of the diagram $*\leftarrow X\to *$. Dually, for X fibrant, the loop sapce ΩX is a model for the homotopy pullback of $*\to X\leftarrow *$.*

Proof. We do the proof for the suspension (the other one is similar) and we do it in $\mathtt{Top}$. Consider the following pushout square

$$\begin{array}{ccc} \{0,1\} & \longrightarrow & [0,1] \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \end{array}$$

We can make this into a pushout square in $\mathtt{Top}_*$ by picking 0 as a base point in the two top spaces. This is also a homotopy pushout square since all objects are cofibrant and the map $\{0,1\} \to [0,1]$ is a cofibration.

Smashing with a cofibrant space is a left Quillen functor, so the square

$$\begin{array}{ccc} X & \longrightarrow & [0,1]\wedge X \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

is also a homotopy pushout square. Since the map $[0,1]\wedge X \to *$ is a weak equivalence, we are done. □

Given a map $f : A \to X$ in $\mathtt{S}_*$. The cone of f is by definition the homotopy pushout of $* \leftarrow A \to X$. This can be modelled as $C(f) = A \times [0,1] \sqcup X/\sim$ where the equivalence relation $\sim$ identifies the points of the form $(a,1)$ to $f(a)$ and identifies all the points of the form $(a,0)$ and $(*,t)$ together.

Recall that an abelian group object in a category with products is simply an object A equipped with a multiplication map $A \times A \to A$ and an unit $* \to A$ satisfying the usual axioms of an abelian group. An important example of abelian group objects in $\mathtt{S}_*$ is given by the following proposition.

PROPOSITION 4.4. *Let X be a fibrant object in $\mathtt{S}_*$, then $\Omega^2 X$ is an abelian group object in $\mathtt{S}_*$.*

PROOF. Exercise. □

PROPOSITION 4.5. *Let M be an abelian group object in the homotopy category of $\mathtt{S}_*$ (for example a two-fold loop space). Then the sequence of maps*

$$A \to X \to C(f)$$

induces an exact sequence of abelian groups

$$[C(f), M] \to [X, M] \to [A, M]$$

PROOF. Assume that M is fibrant. The functor $\mathrm{map}_*(-, M); \mathtt{S}_*^{\mathrm{op}} \to \mathtt{S}$ is a Quillen right functor, it follows that the square

$$\begin{array}{ccc} \mathrm{map}(C(f), M) & \longrightarrow & \mathrm{map}(X, M) \\ \downarrow & & \downarrow \\ \mathrm{map}(*, M) & \longrightarrow & \mathrm{map}(A, M) \end{array}$$

is a homotopy pullback square in $\mathtt{S}_*$. We thus get an exact sequence of pointed sets

$$\pi_0(\mathrm{map}(C(f), M)) \to \pi_0 \mathrm{map}(X, M) \to \pi_0(\mathrm{map}(A, M))$$

or equivalently

$$[C(f), M] \to [X, M] \to [A, M]$$

Since M is an abelian group object in $\mathtt{Ho}(\mathtt{S}_*)$, so is $[Y, M]$ for any pointed space Y. Moreover, a map $Y \to Z$ induces a map of abelian groups $[Z, M] \to [Y, M]$. The result follows. □

COROLLARY 4.6. *Same notations. There is a long exact sequence of abelian groups*

$$[C(f), M] \to [X, M] \to [A, M] \to [C(f), \Omega M] \to [X, \Omega M] \to [A, \Omega M] \to [C(f), \Omega^2 M] \to \dots$$

There is a dual statement given by the following proposition.

PROPOSITION 4.7. *Consider a fiber sequence of fibrant objects $F \to E \to B$ in $\mathtt{S}_*$. Then for any space A, there is a long exact sequence of abelian groups*

$$[A, \Omega^2 F] \to [A, \Omega^2 E] \to [A, \Omega^2 B] \to [A, \Omega^3 F] \to [A, \Omega^3 E] \to [A, \Omega^3 B] \to [A, \Omega^4 F] \to \dots$$

2. Spectra

Definition 4.8. A prespectrum is a sequence of pointed spaces $\{X_n\}$ together with the data of maps $X_n \to \Omega X_{n+1}$. A map of prespectra $X = \{X_n\} \to Y = \{Y_n\}$ is a collection of maps of pointed spaces $X_n \to Y_n$ such that the diagrams

$$\begin{array}{ccc} X_n & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \Omega X_{n+1} & \longrightarrow & \Omega Y_{n+1} \end{array}$$

commute.

Definition 4.9. A spectrum is a prespectrum such that the maps $X_n \to \Omega X_{n+1}$ are weak equivalences. A map of spectra is a map between the underlying prespectra.

Example 4.10.
- For a pointed space X, we denote by $\Sigma^\infty(X)$ the prespectrum whose n-th space is $\Sigma^n(X)$ and in which the map $\Sigma^n X \to \Omega\Sigma^{n+1} X$ is adjoint to the identity map $\Sigma^{n+1} X \to \Sigma^{n+1} X$.
- For A an abelian group, there is a spectrum HA whose n-th space is the Eilenberg-MacLane space $K(A, n)$ (to fix a model we can take the Dold-Kan inverse of the chain complex which is A in degree n and zero elsewhere). The structure maps are simply the weak equivalences $K(A, n) \simeq \Omega K(A, n + 1)$.
- Let U be the infinite unitary group. Then the space $\mathbb{Z} \times BU$ represents complex topological K-theory, i.e. for X a finite CW-complex, there is an isomorphism $K^0(X) \cong [X, \mathbb{Z} \times BU]$. A deep theorem called Bott periodicity asserts that there is a weak equivalence $\mathbb{Z} \times BU \simeq U$. It follows that we can produce a spectrum KU given by $KU_i = \mathbb{Z} \times BU$ if i is even and U if i is odd.

Definition 4.11. Let $E = \{E_n\}$ be a spectrum. Let X be a pointed space, let i be an integer. Then we define the i-th cohomology group of X with coefficients in E as follows.

(1) If $i \geq 0$, then it is $[X, E_i]$.
(2) If $i < 0$, then it is $[X, \Omega^{-i} E_0]$.

We denote it by $\tilde{E}^i(X)$. If X is an unpointed space, we write $E^i(X)$ for $\tilde{E}^i(X_+)$,

Remark 4.12. Observe that we could define uniformly $\tilde{E}^i(X)$ as $[X, \Omega^{-i}E_0]$ if we interpret $\Omega^{-i}E_0$ as E_i if i is positive. This makes sense since $\Omega^i E_i \simeq E_0$.

Observe also that these sets are in fact abelian groups. Indeed, each space E_i is weakly equivalent to a 2-fold loop space since $E_i \simeq \Omega^2 E_{i+2}$

Proposition 4.13. *Let E be a spectrum, the collection of functor $\mathtt{S}_* \to \mathtt{Ab}^{\mathrm{op}}$ given by $X \mapsto \tilde{E}^i(X)$ satisfies the following properties.*

(1) *They send weak equivalences of pointed spaces to isomorphisms.*
(2) *If $f : A \to X$ is any map, then there is a long exact sequence*

$$\ldots \tilde{E}^{i-1}(A) \to \tilde{E}^i(C(f)) \to \tilde{E}^i(X) \to \tilde{E}^i(A) \to \tilde{E}^{i+1}(C(f)) \to \ldots$$

(3) *If $\{X_\alpha\}_{\alpha \in A}$ is a collection of pointed spaces, the natural map*

$$\tilde{E}^i(\bigvee_\alpha X_\alpha) \to \prod_\alpha \tilde{E}^i(X_\alpha)$$

is an isomorphisms.

Proof. This is straightforward. Property (2) follows from Corollary 4.6. □

Example 4.14. If A is an abelian group, the cohomology theory represented by HA is the ordinary cohomology with coefficients in A (in that case, there are no negative cohomology groups). The cohomology theory represented by KU is periodic topological K-theory.

3. The model structure on spectra

3.1. Left Bousfield localization.

Definition 4.15. Let $\mathtt{M}$ be a model category. A left Bousfield localization of $\mathtt{M}$ is a model category structure on $\mathtt{M}$ with the same cofibration but more weak equivalences.

Observe that, the identity functor from a model category to a Bousfield localization is a left Quillen functor (with right adjoint the identity functor). Moreover, we have the following proposition.

Proposition 4.16. *Let $\mathtt{M}$ be a model category and $L\mathtt{M}$ be a Bousfield localization of $\mathtt{M}$, then the right adjoint in the adjunction*

$$\mathbb{L}\mathrm{id} : \mathtt{M} \leftrightarrows L\mathtt{M} : \mathbb{R}\mathrm{id}$$

is fully faithful.

Proof. It is equivalent to prove that the unit of the adjunction is an isomorphism in $\mathtt{Ho}L\mathtt{M}$. Let X be an object of $\mathtt{M}$ that we may assume to be cofibrant (in $\mathtt{M}$ or in $L\mathtt{M}$, this does not make a difference). The unit of this adjunction on X is then simply $X \to R(X)$ where R is a fibrant replacement functor in $L\mathtt{M}$. By definition, this map is a weak equivalence in $L\mathtt{M}$ and hence an isomorphism in the homotopy category. □

Usually, the left Bousfield localizations that we shall consider will be generated by a class of maps that we wish to invert.

Before giving the definition, let us recall that in a simplicial model category, the mapping simplicial set

$$\mathrm{map}(-,-) : \mathtt{M}^{\mathrm{op}} \times \mathtt{M} \to \mathtt{sSet}$$

is a Quillen bifunctor. In particular, it preserves weak equivalences between cofibrant objects in the first variable and weak equivalences between fibrant objects in the second variable. We shall denote by $\mathbb{R}\,\mathrm{map}(A, X)$ the value $\mathrm{map}(QA, RX)$ where $QA \to A$ is a cofibrant replacement and $X \to RX$ is a fibrant replacement.

Proposition 4.17. *Let $\mathtt{M}$ be a simplicial model category and A and X be two obejcts of $\mathtt{M}$, then we have an isomorphism*

$$\pi_0 \mathbb{R}\,\mathrm{map}(A, X) \cong [A, X]$$

Proof. Without loss of generality, we may assume that A is cofibrant and X is fibrant, then the mapping space $\mathrm{map}(A, X)$ is a Kan complex. Its 0-simplices are maps $A \to X$ in $\mathtt{M}$ and two such simplices are identified in π_0 if and only if there is a map

$$\Delta[1] \otimes A \to X$$

whose restriction to both ends of the interval coincide with the two maps. So it is enough to check that $A \sqcup A \to \Delta[1] \otimes A$ is a cylinder for A which is straightforward. □

Definition 4.18. Let $\mathtt{M}$ be a simplicial model category and S be a class of maps in $\mathtt{M}$. We say that an object U of $\mathtt{M}$ is S-local if for any $f : A \to B$ a map in S, the induced map

$$\mathbb{R}\,\mathrm{map}(B, U) \to \mathbb{R}\,\mathrm{map}(A, U)$$

is a weak equivalence. We say that a map $f : P \to Q$ is an S-local weak equivalence if for any S-local object U, the induced map

$$\mathbb{R}\,\mathrm{map}(Q, U) \to \mathbb{R}\,\mathrm{map}(P, U)$$

is a bijection.

Example 4.19.

Fix a positive integer k and consider the set with just one map

$$f : S^k \to *$$

in $\mathtt{Top}_*$. Then, a space (X, x) is local with respect to these maps if and only, for all $n \geq k$, the group $\pi_n(X, x)$ vanishes (Exercise). A map is an S-local weak equivalence if and only if it induces an isomorphism on π_n for $n \geq k$.

We have the following existence result.

THEOREM 4.20 (Smith). *Let* $\mathtt{M}$ *be a presentable category equipped with a cofibrantly generated and simplicial model structure. Let* S *be a set of maps in* $\mathtt{M}$. *Then there is a model structure* $L_S\mathtt{M}$ *on* $\mathtt{M}$ *in which*

(1) *The cofibrations are the cofibrations in* $\mathtt{M}$.
(2) *The weak equivalences are the* S*-local weak equivalences.*
(3) *The fibrant objects are the* S*-local objects that are also fibrant in* $\mathtt{M}$.

We make the following final observation about this model structure.

PROPOSITION 4.21. *Same notations. In the model structure* $L_S\mathtt{M}$, *a map between two* S*-local objects is a weak equivalence if and only if it is a weak equivalence in* $\mathtt{M}$.

PROOF. Indeed, consider the Quillen adjunction

$$\mathtt{M} \leftrightarrows L_S\mathtt{M}$$

Then the right adjoint (which is just the identity functor) preserves weak equivalences between fibrant objects in $L_S\mathtt{M}$. Since any S-local object is weakly equivalent in $\mathtt{M}$ to a fibrant object in $L_S\mathtt{M}$ we are done. □

3.2. The stable model structure on prespectra. Let $\mathtt{PSp}$ be the category of simplicial prespectra. We shall take the convention that the smah product of a pointed simplicial set with a prespectrum is given by

$$K \wedge \{X_n\} := \{K \wedge X_n\}$$

This defines a simplicial structure on $\mathtt{PSp}$ with tensoring $\mathtt{sSet} \times \mathtt{PSp} \to \mathtt{PSp}$ given by

$$K \otimes X := K_+ \wedge X$$

We deduce a mapping simplicial set given by the formula

$$\mathrm{map}(X, Y)_n := \mathrm{Hom}_{\mathtt{PSp}}(\Delta[n] \otimes X, Y)$$

For K a pointed space, we denote by F_nK the prespectrum given by

$$\begin{aligned} F_nK_i &= * \text{ if } i < n, \\ &= \Sigma^{i-n}K \text{ if } i \geq n. \end{aligned}$$

Observe that for $X = \{X_n\}$ a prespectrum, we have a natural isomorphism

$$\mathrm{map}(F_nK, X) \cong X_n$$

PROPOSITION 4.22. *There is a model structure on* $\mathtt{PSp}$ *in which*

(1) *A map* $f : X \to Y$ *is a weak equivalence if* $f_n : X_n \to Y_n$ *is a weak equivalence for each* n.
(2) *A map* $f : X \to Y$ *is a fibration if* $f_n : X_n \to Y_n$ *is a fibration for each* n.

PROOF. There is an adjunction

$$\prod_{n \in \mathbb{N}} \mathtt{sSet}_* \leftrightarrows \mathtt{PSp}$$

in which the right adjoint is the obvious forgetful functor and the left adjoint sends a collection $(K_n)_{n\in\mathbb{N}}$ of pointed spaces to the prespectrum $\bigvee_{n\in\mathbb{N}} F_nK_n$. This is then simply an application fo the transfer theorem (Theorem 2.11). □

This is not the correct model category to model the homotopy theory of spectra. For this, we will have to introduce a certain Bousfield localization of PSp.

THEOREM 4.23. *There is a model structure denoted* Sp *on* PSp *in which*

(1) *The cofibrations are the same as in* PSp.
(2) *The fibrant objects are the spectra that are fibrant in* PSp.
(3) *the weak equivalences are the map* $f : X \to Y$ *such that for any spectrum* U*, the induced map*

$$[Y, U]_{\mathtt{PSp}} \to [X, U]_{\mathtt{PSp}}$$

is a bijection.

PROOF. First, we observe that for any based space K, there is a map of prespectra

$$\Sigma F_{n+1} K \to F_n K$$

which is the unique map $* \to K$ in level n and the indentity map elsewhere.

Then consider the map

$$f_n : \Sigma F_{n+1} S^0 \to F_n S^0$$

It is easy to check that for a prespectrum X, the induced map

$$\mathrm{map}(F_n S^0, X) \to \mathrm{map}(\Sigma F_{n+1} S^0, X)$$

is the structure map

$$X_n \to \Omega X_{n+1}$$

It follows that if we take the Bousfield localizations with respect to these maps, we will obtain a model structure that satisfies (1) and (2). In this model structure, the weak equivalences will be the maps $f : A \to B$ such that, for any spectrum U, the induced map

$$\mathbb{R}\,\mathrm{map}(B, U) \to \mathbb{R}\,\mathrm{map}(A, U)$$

is a weak equivalence. In particular, applying π_0, we recover the property (3). Conversely, we can show that maps satisfying property (3) satisfies the stronger condition above. Indeed, without loss of generality, we may assume that U is fibrant. Then, if we apply π_i to the map

$$\mathbb{R}\,\mathrm{map}(B, U) \to \mathbb{R}\,\mathrm{map}(A, U)$$

we get the map

$$[B, \Omega^i U] \to [A, \Omega^i U]$$

(where Ω^i of a prespectrum is simply levelwise application of the functor Ω^i). Since $\Omega^i U$ is still a fibrant spectrum, this map is a bijection by hypothesis. In the end, the map

$$\mathbb{R}\,\mathrm{map}(B, U) \to \mathbb{R}\,\mathrm{map}(A, U)$$

induces an isomorphism on all the homotopy groups. □

There is a simpler interpretation of the weak equivalences in terms of homotopy groups.

DEFINITION 4.24. For a fibrant prespectrum $X = \{X_n\}$ and $i \in \mathbb{Z}$, the i-th homotopy group is defined by

$$\pi_i(X) = \mathrm{colim}_{n \geq \max(-i,0)} \pi_{i+n} X_n$$

where the map

$$\pi_{i+n} X_n \to \pi_{i+n+1} X_{n+1}$$

is induced by the structure map $X_n \to \Omega X_{n+1}$.

The homotopy group of a general prespectrum are defined as the homotopy groups of a fibrant replacement.

THEOREM 4.25. *In* Sp *the weak equivalences are the* π_**-isomorphisms (i.e. the maps that induce isomorphisms on all homotopy groups).*

PROOF. We shall not prove this but let us make a few observations. First of all, they satisfy the two-out-of-three property. It is clear that a map which is a weak equivalence in PSp is a π_*-isomorphism. Moreover, if a map $f : X \to Y$ between fibrant spectra is a π_*-isomorphism then, it is a weak equivalence (in PSp or Sp it does not matter in this case thanks to Proposition 4.21). Indeed, for a fibrant spectrum, we have $\pi_i(X_n) \cong \pi_{i-n}(X)$. It follows that between fibrant spectra, the π_*-isomorphism are exactly the same as the weak equivalences. □

There is an explicit fibrant replacement in Sp.

CONSTRUCTION 4.26. Let X be a fibrant prespectrum, we define QX_n to be the colimit

$$QX_n = \operatorname{colim}_i \Omega^i X_{n+i}$$

where the colimit is defined using the maps

$$\Omega^i X_{n+i} \to \Omega^{i+1} X_{n+i+1} = \Omega^i \Omega X_{n+i+1}$$

given by the i-th loop of the structure map. There is an obvious isomorphism

$$QX_n \to QX_{n+1}$$

as the two colimits diagrams are the same except that in the target we start one step later (but this does not affect the colimit). It follows that the spaces QX_n assemble into a spectrum. Moreover, it is still levelwise fibrant as Kan complexes are stable under filtered colimits.

The map $X \to QX$ is clearly a π_*-isomorphism. It follows that $X \to QX$ is a fibrant replacement in Sp. If X was not levelwise fibrant, a fibrant replacement would be given by $X \to QX'$ with $X \to X'$ a fibrant replacement in PSp.

We deduce an explicit description of the Hom sets in HoSp whose source is an infinite suspension.

PROPOSITION 4.27. *Let K be a based simplicial set and E be a prespectrum. Then*

$$[\Sigma^\infty K, E]_{\mathtt{Sp}} \cong \operatorname{colim}_n [\Sigma^n K, E_n]_{\mathtt{sSet}_*}$$

where the colimit is taken along the maps

$$[\Sigma^n K, E_n] \xrightarrow{\Sigma} [\Sigma^{n+1} K, \Sigma E_n] \xrightarrow{\textit{structure map}} [\Sigma^{n+1} K, E_{n+1}]$$

PROOF. Exercise. □

For a prespectrum $X = \{X_n\}_{n\in\mathbb{N}}$ and k an integer, we denote by $s^k X$ the prespectrum given by

$$(s^k X)_n = X_{n+k}$$

where we take the convention that $X_n = *$ if $n < 0$. We also have the suspension and loop space fucntor in PSp obtained by levelwise application of the suspension and loop functor of $\mathtt{sSet}_*$.

PROPOSITION 4.28. *We have*

(1) *The obvious maps $s^1 s^{-1} X \to X$ and $X \to s^{-1} s^1 X$ are weak equivalences in* Sp.
(2) *The map $s^{-1} X \to \Omega X$ induced by the structure maps $X_n \to \Omega X_{n+1}$ is a weak equivalence in* Sp.
(3) *The map $\Sigma X \to s^1 X$ induced by the structure maps $\Sigma X_n \to X_{n+1}$ is a weak equivalence in* Sp.

PROOF. (1) First observe that s^1 is left adjoint to s^{-1} and the obvious maps are just the counits and units of this adjunction. Clearly the counit $s^1 s^{-1} X \to X$ is an isomorphism. The unit is an isomorphism in all level except zero but this means that it is a π_*-isomorphism.

(2) This map is obviously a π_*-isomorphism.

(3) First observe that both Σ and s^1 are left Quillen functors from PSp to PSp, so they induce left adjoint in the homotopy category. Let E be a spectrum, we have

$$[s^1 X, E] \cong [X, s^{-1} E] \cong [X, \Omega E] \cong [\Sigma X, E]$$

where the first and last isomorphism comes from the adjunction and the second from part (2). Morever, one check easily that this map coincides with the one coming from $\Sigma X \to s^1 X$. It follows that this map is a weak equivalence as desired. □

The upshot of this Proposition is that in the model category $\mathtt{Sp}$, the loop and suspension functor coincides up to weak equivalences with the two shift functors and induce mutually inverse endofunctors of the homotopy category.

4. Gamma-spaces

We denote by $\langle n\rangle$ the finite pointed set $\{0, \ldots, n\}$ pointed by zero.

Definition 4.29. We denote by Γ the opposite of the category whose objects are the sets $\langle n\rangle$ and morphisms are maps of finite pointed sets.

For $i \in \{1, \ldots, n\}$, we denote by $\rho^i : \langle n\rangle \to \langle 1\rangle$ the map that sends i to 1 and all other elements to zero.

Proposition 4.30. *There is an equivalence of categories between the category of abelian monoids and the category of functor Γ^{op} to sets that satisfy the following conditions.*

1. *The canonical map $X\langle 0\rangle \to *$ is an isomorphism.*
2. *The map $X\langle n\rangle \to X\langle 1\rangle^n$ induced by the maps ρ^i is a bijection.*

Proof. For an abelian monoid A, define a functor F_A from γ^{op} to sets by the formula

$$F_A\langle n\rangle = A^n$$

and given $f : \langle n\rangle \to \langle m\rangle$, the map

$$F_A(f) : A^n \to A^m$$

sends an n-tuple $(a_i)_{1\leq i\leq n}$ to $(\sum_{f(i)=j} a_i)_{1\leq j\leq m}$. One easily checks that this functor satisfies condition (1) and (2). Conversely, given a functor that satisfies (1) and (2), we can give an abelian monoid structure to $X\langle 1\rangle$ by declaring the composition to be

$$X\langle 1\rangle^2 \cong X\langle 2\rangle \xrightarrow{X(\mu)} X\langle 1\rangle$$

where $\mu : \langle 2\rangle \to \langle 1\rangle$, is the map that sends 1 and 2 to 1. □

Definition 4.31. A functor $X : \Gamma^{\mathrm{op}} \to \mathtt{sSet}_*$ is called special if

1. The canonical map $X\langle 0\rangle \to *$ is a weak equivalence.
2. The map $X\langle n\rangle \to X\langle 1\rangle^n$ induced by the maps ρ^i is a wek equivalence.

From the previous proposition, we see that if X is a special Γ-space, the set $\pi_0 X\langle 1\rangle$ is naturally equipped with an abelian monoid structure.

Proposition 4.32. *We say that a Γ-space is very special if it is special and $\pi_0 X\langle 1\rangle$ is an abelian group.*

Construction 4.33. Let K be a pointed simplicial set which is degreewise finite. Then, we can view K as a functor $K : \Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$. For a Γ-space X, we define $X(K)$ to be $|X \circ K|$.

Proposition 4.34. *Consider the category whose objects are the totally ordered sets $[n]$ with $[n] \geq 1$ and morphisms are the order preserving maps that preserve the minimal and maximal element. Then this category is isomorphic to Δ^{op}.*

Proof. Let us call this category I. There is a functor $\alpha : I \to \Delta^{\mathrm{op}}$. This functor sends $[n]$ to $[n-1]$. We think of $\alpha[n]$ as the n intervals $[i, i+1]$ for $i \in \{0, \ldots, n-1\}$. Then given a map $f : [n] \to [m]$ preserving both end points, the map $\alpha(f) : [m-1] \to [n-1]$ sends the interval $[i, i+1]$ to the unique interval $[j, j+1]$ such that $[i, i+1] \subset [f(j), f(j+1)]$. One easily checks that α is indeed an equivalence of categories $I \simeq \Delta^{\mathrm{op}}$. □

With this description, there is a functor $\Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ given on objects by the formula

$$[n] \mapsto \langle n-1 \rangle$$

and sending a map $f : [m] \to [n]$ to the map $g : \langle m-1 \rangle \to \langle n-1 \rangle$ given by $g(i) = f(i)$ if $f(i) \neq n$ and $g(i) = 0$ if $f(i) = n$.

LEMMA 4.35. *This functor $\Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ viewed as based simplicial set is isomorphic to $\Delta[1]/\partial\Delta[1]$.*

CONSTRUCTION 4.36. let $A : \Gamma^{\mathrm{op}} \to \mathtt{sSet}$ be a Gamma-space. We extend A to a functor $\mathtt{Set}_*$ by the formula

$$A(S) = \mathrm{colim}_{f:\langle n \rangle \to S} A\langle n \rangle$$

We extend A further to $\mathtt{sSet}_*$ by the formula

$$A(K) = |[n] \mapsto A(K_n)|$$

CONSTRUCTION 4.37. Observe that, for K and L two based simplicial sets there is a natural map $K \wedge A(L) \to A(K \wedge L)$. We can thus extend A to a functor from $\mathtt{PSp}$ to $\mathtt{PSp}$ by the formula

$$A(E)_n = A(E_n)$$

with structure map given by

$$S^1 \wedge A(E_n) \to A(S^1 \wedge E_n) \to A(E_{n+1})$$

PROPOSITION 4.38. *If A is a (simplicial) abelian group viewed as a Γ-space, then $A(S^1)$ is exactly the classifying space $B(A)$. In particular, we have*

$$\pi_i(A(S^1)) \cong \pi_{i-1}(A)$$

PROOF. If A is a discrete abelian group, we can identify the simplicial object $[n] \mapsto A(S^1_n)$ as the bar construction $B_\bullet(*, A, *)$. If A is a simplicial abelian group, we shall admit this fact. □

In fact, we have the following more general theorem.

THEOREM 4.39. *If A is a very special Γ-space, then the prespectrum $A(\mathbb{S})$ is a spectrum.*

PROOF. We shall admit this fact. This is Theorem 4.2 in Bousfield-Friedlander's paper "Homotopy theory of Gamma-spaces, spectra and bisimplicial sets". □

EXAMPLE 4.40. In particular, if A is a discrete (or simplicial) abelian group, we obtain a spectrum HA with $HA \cong B^n A$ (the n-fold bar construction of A).

The homotopy theory of Γ-spaces can be studied via model categories.

PROPOSITION 4.41. *There is a model structure on Γ-spaces such that*

(1) *The cofibrations are the projective cofibrations.*
(2) *The fibrant objects are the projectively fibrant Γ-spaces that are special (resp. very special).*
(3) *The weak are the maps $f : A \to B$ such that for any special (resp. very special) Γ-space X, the induced map*

$$\mathbb{R}\,\mathrm{map}(B, X) \to \mathbb{R}\,\mathrm{map}(A, X)$$

is a weak equivalence.

PROOF. We start from the projective model structure and find a set of maps such that the local objects are the special or very special Gamma-spaces. □

THEOREM 4.42. *A map $f : A \to B$ is a weak equivalence in the very special model structure if and only if the induced map $A(\mathbb{S}) \to B(\mathbb{S})$ is a weak equivalence in $\mathtt{Sp}$. Moreover, the functor $A \mapsto A(\mathbb{S})$ induces an equivalence of categories from the homotopy category of very special Γ-spaces to the homotopy category of connective spectra.*

We have the following important proposition.

Proposition 4.43. *Let A be a special Γ-space. Then $A(S^1 \wedge -)$ is a very special Γ-space. Moreover, the map*

$$S^1 \wedge A(-) \to A(S^1 \wedge -)$$

is adjoint to a map

$$A(-) \to \Omega A(S^1 \wedge -)$$

which is a weak equivalence in the very special model structure on $\mathtt{sSet}_*^{\Gamma^{\mathrm{op}}}$.

4.1. Construction of Γ-spaces. Let C be a simplicial category with finite coproducts. Then C is not quite a strictly commutative object in $\mathtt{Cat}$, however, we may construct a Γ-category with $C\langle 1\rangle = C$. In order to do this, we define $C\langle m\rangle$ to be the category whose objects are pairs $(f, c_1, \dots, c_n)$ with $f : \langle n\rangle \to \langle m\rangle$ a map in Γ^{op} and $c_1, \dots, c_n$ an n-tuple of objects of C. A morphism from $(f, c_1, \dots c_n)$ to $(f', c'_1, \dots, c'_{n'})$ is a morphism in C^m:

$$(\oplus_{f(i)=1} c_i, \dots, \oplus_{f(i)=m} c_i) \to (\oplus_{f'(i)=1} c'_i, \dots, \oplus_{f'(i)=m} c'_i)$$

The assignment $\langle n\rangle \to C\langle n\rangle$ is a functor from Γ^{op} to $\mathtt{Cat}$. Moreover, there is an equivalence of categories $C\langle n\rangle \to C^n$ sending $(f, c_1, \dots, c_n)$ to $(\oplus_{f(i)=1} c_i, \dots, \oplus_{f(i)=m} c_i)$. It is in fact not difficult to prove the following

Proposition 4.44. *The assignment $\langle n\rangle \mapsto C\langle n\rangle$ is a special Γ-cateogry. That is, the collection of maps ρ^i induce an equivalence of categories*

$$C\langle n\rangle \to C^n$$

Construction 4.45. Given a simplicial category with finite coproducts C, we define a functor

$$N(C) : \Gamma^{\mathrm{op}} \to \mathtt{sSet}$$

by sending $\langle n\rangle$ to $N(C\langle n\rangle)$ where the nerve of a simplicial category C is simply the diagonal of the simplicial simplicial set

$$[n] \mapsto \coprod_{c_0, \dots, c_n} \mathrm{Hom}_C(c_0, c_1) \times \dots \times \mathrm{Hom}_C(c_{n_1}, c_n)$$

Proposition 4.46. *The Γ-space $N(C)$ is special.*

Proof. This simply comes from the fact that the nerve functor preserves products and sends equivalence of categories to weak equivalences. □

We can thus denote by $C \mapsto K(C)$ a fibrant replacement of C in the

Example 4.47.
(1) Algebraic K-theory.
(2) Topological K-theory.
(3) The sphere spectrum.

CHAPTER 5

Introduction to chromatic homotopy theory

Let $\mathtt{Sp}$ be a point-set level model for spectra, we assume that $\mathtt{Sp}$ has a good notion of smash product. The purpose of stable homotopy theory is to study the homotopy category $\mathtt{HoSp}$ of $\mathtt{Sp}$. We are in fact happy to restrict to $\mathtt{HoSp}_f$, the homotopy category of finite spectra (i.e., spectra that can be constructed by glueing finitely many cells).

1. Generalized homology theories and Hopf algebroid

The general method for attacking this problem is to use functors from $\mathtt{HoSp}$ to purely algebraic categories. To each monoid object E of $\mathtt{Sp}$, we can assign a functor

$$E_* : \mathtt{HoSp} \to \mathtt{Ab}_*$$

sending X to $E_*(X) = \pi_*(E \wedge X)$.

This functor has more structure than just an abelian group. First of all, $E_* = E_*(\mathbb{S})$ is a graded ring and $E_*(X)$ is a left E_*-module functorially in X. Indeed, let $u : \mathbb{S}^k \to E \wedge X$ and $v : \mathbb{S}^l \to E$. We can smash them together and compose with the multiplication in E

$$\mathbb{S}^{l+k} \xrightarrow{v \wedge u} E \wedge (E \wedge X) \to E \wedge X$$

taking homotopy groups, this gives a pairing

$$E_k(X) \otimes E_l \to E_{k+l}(X)$$

In fact, we will see that this functor often has a lot more structure. We assume that E is connective, that E_*E is flat as an E_*-module and that E is a monoid in $\mathtt{Sp}$ such that E_* is commutative (we do not need to assume that E is commutative in $\mathtt{Sp}$ and it is often useful to not have to assume that).

Remark 5.1. If X and Y are any spectra, there is a spectral sequence

$$\mathrm{Tor}^{E_*}_{s,t}(E_*X, E_*Y) \implies E_{s+t}(X \wedge Y)$$

Our flatness assumption implies that for any spectrum X, $E_*(X \wedge E)$ is isomorphic to $E_*X \otimes_{E_*} E_*E$.

If this assumption is satisfied, we have several maps relating E_* and E_*E.

- The multiplication map
$$\mu : E_*E \to E_*$$
- The two unit maps
$$\eta_R, \eta_L : E_* \to E_*E$$
- The comultiplication map
$$c : E_*E \to E_*E \otimes_{E_*} E_*E$$
this is induced from the map $E \wedge \mathbb{S} \wedge E \to E \wedge E \wedge E$ by taking homotopy groups. We need flatness of E_*E over E_* to make this work.
- The antipode
$$\tau : E_*E \to E_*E$$
induced from the map $E \wedge E \to E \wedge E$ swapping the two factors.

These maps satisfy a bunch of compatibility. The best way to express them is via the following definition:

DEFINITION 5.2. A Hopf-algebroid is a pair (A, Γ) of (graded) commutative algebras which is a cogroupoid object in commutative algebras. That is, for any commutative algebra R, there is a groupoid whose objects is $\mathrm{Hom}(A, R)$ and whose morphisms is $\mathrm{Hom}(\Gamma, R)$ and which depends functorially on R.

If we unwrap this definition using Yoneda's lemma, we see that the pair (A, Γ) must be equipped with

- a map
$$\mu : \Gamma \to A$$
which is dual to the "identity" maps $\mathrm{Ob} \to \mathrm{Mor}$.
- Two maps $\eta_R, \eta_L : A \to \Gamma$ corresponding to the source and target maps $\mathrm{Mor} \to \mathrm{Ob}$.
- A map $c : \Gamma \to \Gamma \sqcup_A \Gamma = \Gamma \otimes_A \Gamma$ dual to the composition map $\mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor}$.
- A map $\Gamma \to \Gamma$ corresponding to the inverse map $\mathrm{Mor} \to \mathrm{Mor}$.

All these maps have to satisfy a bunch of axioms corresponding to the axiom verified by a groupoid. Now we can state the following proposition:

PROPOSITION 5.3. *The pair (E_*, E_*E) equipped with the maps η_R, η_L, μ, c and τ is a Hopf algebroid.*

DEFINITION 5.4. A left comodule over a Hopf algebroid (A, Γ) is an A-module M, equipped with a map
$$\rho : M \to \Gamma \otimes_A M$$
such that the composite
$$M \xrightarrow{\rho} \Gamma \otimes_A M \xrightarrow{\mu \otimes_A M} M$$
is the identity

and the two maps $(\Gamma \otimes_A \rho) \circ \rho$ and $(c \otimes_A M) \circ \rho$:
$$M \to \Gamma \otimes_A \Gamma \otimes_A M$$
are the same.

REMARK 5.5. Note that Γ is an A-module in two different ways using the two units η_R and η_L. We make the conventian that any time we write $\Gamma \otimes_A -$, Γ is an A-module via η_R and any time we write $- \otimes_A \Gamma$, Γ is an A-module via η_L.

PROPOSITION 5.6. *If X is a spectrum, then $E_*(X)$ is a comodule over (E_*, E_*E).*

PROOF. The structure map is given by applying E_* to
$$\mathbb{S} \wedge X \to E \wedge X$$
The fact that this makes E_*X into an E_*E-comodule is straightforward. □

EXAMPLE 5.7. Take $E = H\mathbb{Q}$. Then $E_*E = \mathbb{Q}$. The Hopf algebroid is trivial $(\mathbb{Q}, \mathbb{Q})$.

EXAMPLE 5.8. Take $E = H\mathbb{F}$ for $\mathbb{F}$ a finite field with p elements. Then $E_*E = \mathscr{A}_*$ is the dual Steenrod algebra. The Hopf algebroid we get is in fact a Hopf algebra over $\mathbb{F}$. We recover the classical fact that $H\mathbb{F}_*X$ is a comodule over $\mathscr{A}_*$.

In general, we have lifted our functor $\mathtt{Sp} \to \mathtt{Mod}_{E_*}$ to a functor $\mathtt{Sp} \to \mathtt{Comod}_{E_*E}$.

The category of E_*E-comodule is an abelian category with enough injectives. We want to think of the derived category of $\mathtt{Comod}_{E_*E}$ as an approximation for $\mathtt{HoSp}$. A way to detect how unreasonable this is is via the generalized Adams spectral sequence. For a (finite) spectrum X, this is a spectral sequence whose E_2-page is
$$\mathrm{Ext}_{\mathtt{Comod}_{E_*E}}(E_*, E_*X)$$
and which (in good cases) converges to π_*L_EX (where L_EX denotes the E-localization of X). Thus, $\mathtt{Comod}_{E_*E}$ is a reasonable approximation if

- The object L_EX is interesting.
- The spectral sequence is "close" to collapsing.

EXAMPLE 5.9. For instance, if $E = H\mathbb{Q}$, the spectral sequence E_2-page reduces to

$$\mathrm{Hom}(\mathbb{Q}, H_*(X, \mathbb{Q}))$$

in particular it collapes. Unfortunately, the target $\pi_* L_{\mathbb{Q}} X$ was not particularly interesting.

On the other hand if $E = H\mathbb{F}_p$, we recover the classical Adams spectral sequence. The target is very interesting but the spectral sequence is quite far from collapsing.

2. Complex oriented cohomology theories

DEFINITION 5.10. A multiplicative cohomology theory E^* is complex oriented if it has the following data:

For any $\xi \to X$ a complex vector bundle of dimension n, there is a Thom class U_ξ in $\tilde{E}^{2n}(X^\xi)$ such that for any $x \in X$, the composite

$$\tilde{E}^{2n}(X^\xi) \to \tilde{E}^{2n}(\star^\xi) \cong \tilde{E}^{2n}(S^{2n}) \to E^0$$

sends U_ξ to 1. We further require that $U_{f^*\xi} = f^* U_\xi$ and that $U_{\xi \oplus \eta} = U_\xi U_\eta$.

Universal example. Take $MU(n)$ to be the Thom space of the universal $U(n)$-bundle over $BU(n)$. It suffices to define a class in $\tilde{E}^{2n}(MU(n))$ for each n.

EXAMPLE 5.11. The spectrum MU itself is complex oriented. We have $\pi_0(MU) = \mathbb{Z}$ which implies that we have a map $MU \to H\mathbb{Z}$ defining a complex orientation.

The spectrum KU is complex oriented, U_ξ is given by $u^n([\xi] - 1)$ where $u \in KU^2$ is the Bott class.

Take the universal line bundle over $\mathbb{C}P^\infty$. Then if E is complex oriented, we get a Thom class x in $\tilde{E}^2(\mathbb{C}P^\infty)$. This can also be called the first Chern class.

PROPOSITION 5.12. *If E is complex oriented, we have $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$ and also $E^*((\mathbb{C}P^\infty)^2) \cong E^*[[x, y]]$.*

PROOF. This is proved using the Atiyah Hirzebruch spectral sequence. □

There is a multiplicative structure on the set of equivalences classes of line bundles given by the tensor product. This is reflected in the fact that $\mathbb{C}P^\infty$ is a H-space. The multiplication induces a ring map

$$E^*[[x]] \to E^*[[x, y]]$$

which is entirely determined by the image of x.

In integral cohomology, the first Chern class has a nice property with respect to the multiplicative structure that $c_1(\lambda \otimes \mu) = c_1(\lambda) + c_1(\mu)$. In other words, the map

$$\mathbb{Z}[[x]] \to \mathbb{Z}[[x, y]]$$

is the unique ring map sending x to $x + y$.

For a general complex oriented theory this does not hold and the multiplication on $\mathbb{C}P^\infty$ gives us a power series in two variables $F(x, y)$ with coefficients in E^*. This power series is not arbitrary. There are restrictions induced by the fact that the H-space multiplication on $\mathbb{C}P^\infty$ is unital associative and commutative.

PROPOSITION 5.13. *The power series F has to satisfy*

- $F(x, F(y, z)) = F(F(x, y), z)$
- $F(x, 0) = x$
- $F(x, y) = F(y, x)$

DEFINITION 5.14. A power series in two variables with coefficients in R satisfying these two conditions is called a formal group law over R. When R is graded, we further require that F is homogeneous of degree 2 when x and y are given degree 2

PROPOSITION 5.15. *Formal group laws have inverses. That is, if F is a formal group law, there is a power series $i(x)$ such that*

$$F(x, i(x)) = 0$$

EXAMPLE 5.16. The additive formal group law $F(x, y) = x + y$.
The multiplicative formal group law $F(x, y) = x + y + xy$.

There is in fact a groupoid of formal group laws over R.

DEFINITION 5.17. Let F and G be two formal group laws over R. Then a strict isomorphism from F to G is a power series $u(x) = x +$ higher order terms such that

$$u(F(x, y)) = G(u(x), Gu(y))$$

The composition of strict isomorphisms is just the composition of power series.

Note that strict isomorphism are indeed isomorphisms in the sense that they have an inverse. More generally any power series of the form $x+$ higher order terms has an inverse for the composition of power series.

EXAMPLE 5.18. Let R be a commutative $\mathbb{Q}$-algebra. Then we can construct an isomorphism from the additive to the multiplicative formal group law by the exponential formula

$$x + x^2/2 + x^3/6 + \ldots + x^n/n! + \ldots$$

In fact, over $\mathbb{Q}$, all formal group law are isomorphic to the additive formal group law. This is not the case over a field of characteristic p. In that case, the additive and multiplicative formal group laws are never isomorphic.

3. The Lazard ring and Quillen's theorem

The functor from commutative rings to sets sending R to the set of formal group laws over R is representable. This is easy. Just take a giant polynomial algebra $\mathbb{Z}[a_{i,j}, i, j \geq 0]$ and kill everything that prevents

$$F(x, y) = \sum_{i,j} a_{i,j} x^i y^j$$

from being a formal group law. If we declare $a_{i,j}$ to have degree $2 - 2i - 2j$, we get a graded commutative ring L^*.

THEOREM 5.19 (Lazard). *The ring L is a free polynomial algebra $L^* = \mathbb{Z}[x_1, x_2, \ldots]$ where x_i has degree $-2i$.*

We can define as well the universal isomorphism. We have a ring $SI = L[t_1, t_2, \ldots]$ with $|t_i| = -2i$. A map from SI to a commutative ring is the data of a map $L \to R$ (i.e., a formal group law in R) and a map $\mathbb{Z}[t_1, t_2, \ldots] \to R$ which we think of as the data of a power series $x + t_1 x^2 + t_2 x^3 + \ldots$ in R. In other word, the data of a map $SI \to R$ is the same as the data of a formal group law F and an isomorphism u from F to $uF(u^{-1}(x), u^{-1}(y))$. A better formulation of this is that the pair (L, SI) is a Hopf algebroid, representing the functor which to a commutative ring R assigns the groupoid of formal group laws over R and isomorphisms between them.

Since MU is complex oriented, there is a map $L^* \to MU^*$ classifying the formal group law over MU^*.

From now on, we will switch back to the homological grading, $L_k = L^{-k}$, $MU^k = MU_{-k}$ and so on.

THEOREM 5.20 (Quillen). *The map $L_* \to MU_*$ is an isomorphism. Even better, MU_*MU is flat over MU_* and the Hopf algebroid (MU_*, MU_*MU) is isomorphic to (L_*, SI_*).*

4. The height filtration

The idea of chromatic homotopy theory is to approximate a spectrum X by $MU_*(X)$ see as an object in the derived category of MU_*MU-comodules.

Definition 5.21. Let n be a positive integer, and F be a formal group law over a commutative ring R. Then, we define the n-series of F inductively by the formulas

$$[1]_F(x) = x$$
$$[n]_F(x) = F(x, [n-1]_F(x))$$

Proposition 5.22. *Let R be a ring of characteristic p, and F be a formal group law over R, then the p-series of F is either 0 or of the form*

$$[p](x) = \lambda x^{p^n} + \dots$$

for some n and some $\lambda \neq 0$.

Definition 5.23. In the situation of the previous propostion , we say that F has infinite height if the p-series is 0. Otherwise we say that F has height at least n. If λ is invertible, then we say that F has height exactly n.

Example 5.24. Over $\mathbb{F}_p$, we see that the height of the additive formal group law is infinite. The height of the multiplicative formal group law is 1, indeed, we have

$$[p](x) = (1+x)^p - 1 = x^p$$

In particular, the additive and multiplicative formal group law are not isomorphic.

What are the "points" of the Hopf algebroid MU_*MU. In algebraic geometry a point is a map from an algebraic closed field.

Proposition 5.25. *Over an algebraically close field of characteristic 0, any two formal groups are uniquely isomorphic. Over an algebraically closed field of characteristic p, the isomorphism types of formal group laws are in one-to-one correspondance with the integers via the height.*

In other words, if K is of characteristic 0, the groupoid $FGL(K)$ is a point without automorphisms, if K is algebraically closed of characteristic p, $FGL(K)$ splits as a sum of connected groupoids indexed by the positive integers (including ∞). The automorphisms of a the height n formal group law over $\bar{\mathbb{F}}_p$ is called the Morava stabilizer group.

We denote by v_n, the coefficients of x^{p^n} in the $[p]$-series of the universal formal group law over MU_*.

Definition 5.26. We say that M a finitely presented MU_*MU comodule is of type at least n if $M \otimes_{MU_*} MU_*[v_k^{-1}] = 0$ for all $k < n$.

We say that X in $\mathtt{HoSp}_f$ has type at least n if $MU_*(X)$ has type at least n.

What is the "geometry" of $\mathtt{HoSp}_f = D^b(\mathbb{S})$. Can we understand the topological space $Spec(\mathbb{S})$. Amazingly, we can.

Theorem 5.27 (Hopkins–Neeman). *For a Noetherian commutative ring R, we have an isomorphism between the poset of thick subcategories of $D^b(R)$ and the poset of subsets of $Spec(R)$ that are stable under specialization.*

Stable under specialization means that if $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{p}$ is in V, then $\mathfrak{q}$ is in V.

Theorem 5.28 (Devinatz–Hopkins). *The thick subcategories of $\mathtt{HoSp}_f$ are exactly the categories of spectra of type at least n for some prime p and some integer n.*

In other word, the stable homotopy category has a nice stratification and this structure can be detected by the functor

$$\mathtt{HoSp}_f \to D^b(\mathtt{Comd}_{MU_*MU})$$

Contents