MOTIVIC HOMOLOGICAL STABILITY OF CONFIGURATION SPACES

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Abstract. We prove that some of the classical homological stability results for configuration spaces of points in manifolds can be lifted to motivic cohomology.

1. Introduction

A classical result of McDuff and Segal states that the unordered configuration spaces of a connected, open manifold M are homologically stable. More precisely, let M be a connected manifold homeomorphic to the interior of a manifold with non-empty boundary and write

$$C_n(M) = \{(x_1, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}/\Sigma_n$$

for the *unordered configuration space* of n points in M. The theorem of McDuff and Segal (reproven more recently by Randal-Williams) is the following.

Theorem 1.1 ([Seg73, McD75, Seg79, RW13]). *There are stabilization maps of the form $C_n(M) \to C_{n+1}(M)$ such that the induced maps on integral homology*

$$\mathrm{H}_i(C_n(M); \mathbf{Z}) \longrightarrow \mathrm{H}_i(C_{n+1}(M); \mathbf{Z})$$

are isomorphisms when $n \geq 2i$.

This is part of a more general phenomenon of homological stability that holds for many other families of spaces or groups, including general linear groups [vdK80], mapping class groups of surfaces [Har85], automorphism groups of free groups [Hat95, HV98] and moduli spaces of high-dimensional manifolds [GRW17, GRW18].

The goal of this paper is to lift Theorem 1.1 to a version for étale and motivic cohomology. If X is a smooth scheme over a number field, we denote by $\mathrm{Conf}_n(X)$ its associated unordered configuration scheme (described in §2). When X is not complete (and under a minor additional assumption), we construct in §4 stabilization maps

$$M(\mathrm{Conf}_n(X)) \longrightarrow M(\mathrm{Conf}_{n+1}(X)) \tag{1.1}$$

in the category of motives, whose Betti realizations agree with the classical stabilization maps for the unordered configuration spaces of the complex manifold $M = X_{an}$ consisting of the complex points of X. Our main result is then the following.

Theorem 1.2 (Theorems 5.1 and 6.7). *Let X be as above and assume that its étale motive $M_{et}(X)$ is mixed Tate and that the complex manifold X_{an} is connected. Then the maps of étale motivic cohomology groups*

$$\mathrm{H}^{p,q}_{et}(\mathrm{Conf}_{n+1}(X); \Lambda) \longrightarrow \mathrm{H}^{p,q}_{et}(\mathrm{Conf}_n(X); \Lambda)$$

induced by (1.1) are isomorphisms for $p \leq n/2$ and any coefficient ring Λ. In the case when $X = \mathbf{A}^d$ is affine space, the maps of motivic cohomology groups

$$\mathrm{H}^{p,q}(\widetilde{\mathrm{Conf}}_{n+1}(\mathbf{A}^d); \Lambda) \longrightarrow \mathrm{H}^{p,q}(\widetilde{\mathrm{Conf}}_n(\mathbf{A}^d); \Lambda)$$

induced by (1.1) are isomorphisms for $p \leq n/2$ and any coefficient ring Λ.

The statement for étale motivic cohomology is proven in §5 as a direct consequence of the topological case (Theorem 1.1), a detection result for Betti realization (Theorem 3.7) and the existence of the stabilization maps at the étale motivic level. The proof of the statement for motivic cohomology in the case $X = \mathbf{A}^d$ is proven in §6, and is more indirect, since the analogous detection result does not hold in this case. Instead, we use a detection result for the associated graded of the weight filtration, and the key topological input (Lemma 6.8) is a certain twisted homological stability result for the symmetric groups.

Let us emphasize that the first part of the theorem applies to $X = \mathbf{A}^d$ and gives us an étale motivic cohomological stability result. This is different from the second part of the theorem which is a cohomological stability result for motivic cohomology (as opposed to étale motivic cohomology). The price to pay for this finer result is that we have to work with $\widetilde{\mathrm{Conf}}_n$ instead of Conf_n. The configuration space $\widetilde{\mathrm{Conf}}_n$ is a stacky version of the unordered configuration space (see §2 for a precise definition).

Remark 1.3. When X is the affine line $\mathbf{A}^1$, the first statement of Theorem 1.2 (for étale motivic cohomology) has previously been proven in [Hor16],[1] except that the isomorphisms (in a stable range) of [Hor16] are induced not by motivic lifts of *stabilization maps*, but rather by motivic lifts of *scanning maps* $C_n(\mathbf{C}) \to \Omega^2_n S^2$ and of maps between the different path-components of $\Omega^2 S^2$. The paper [Hor16] also uses a different model (denoted by C_n in [Hor16, §5]) for the configuration scheme $\mathrm{Conf}_n(\mathbf{A}^1)$.

Throughout this paper, we denote by K a number field equipped with an embedding $K \to \mathbb{C}$ and we denote by S the spectrum of K. For a smooth scheme X over S, we denote by X_{an} the set $X(\mathbb{C})$ with its complex manifold structure.

2. Quotient and stacky quotient

Let X be a smooth scheme over S and G be a finite group acting on the right on X. We denote by X/G the quotient of X by G in the category of smooth schemes whenever it exists. We denote by $[X/G]$ the "stacky quotient" of X by G. This is a simplicial object in the category of smooth schemes given by

$$[n] \mapsto X \times G^n.$$

This is the nerve of the translation groupoid of the action of G on X.

By mapping a fixed smooth scheme into a simplicial scheme, we can turn a simplicial scheme into a simplicial presheaf on the category of smooth schemes. We will use the same notation to denote a simplicial scheme and the associated simplicial presheaf. Observe that there is a canonical map of simplicial scheme $[X/G] \to X/G$ (where we view X/G as a constant simplicial scheme).

Proposition 2.1. *Assume that the quotient X/G exists in the category of smooth schemes and that the map $X \to X/G$ is an étale map. Then the canonical map $[X/G] \to X/G$ is an étale weak equivalence of simplicial presheaves.*

Proof. Let F be a fibrant object in simplicial presheaves with the Jardine model structure. We denote by f the map induced by precomposition with the canonical map

$$f : \mathrm{Map}(X/G, F) \to \mathrm{Map}([X/G], F).$$

[1] In fact, [Hor16] also claimed to prove the second statement of Theorem 1.2 (for motivic cohomology) for the affine line $X = \mathbf{A}^1$, but the proof contained an error; see the erratum [Hor].

We need to prove that f is a weak equivalence of simplicial sets. The domain of f is simply $F(X/G)$ while the codomain of f is the homotopy limit of the cosimplicial diagram

$$[n] \mapsto F(X \times G^n)$$

Now we observe that there is an isomorphism

$$X \times_{(X/G)} X \times_{(X/G)} \ldots \times_{(X/G)} X \cong X \times G^n$$

where the fiber product on the left has $n+1$ terms and this isomorphism is compatible with the cosimplicial structures on both sides. So the map f can be identified with the map

$$F(X/G) \to \operatorname{holim}_{\Delta}([n] \mapsto F(X^{\times_{(X/G)}(n+1)})).$$

The latter map is a weak equivalence since F is fibrant and $X \to X/G$ is an étale cover. $\square$

For X a smooth scheme over S, we denote by $\mathrm{PConf}_n(X)$ the complement in X^n of the fat diagonal. The Σ_n-action on X^n restricts to a Σ_n-action on $\mathrm{PConf}_n(X)$. We denote by $\mathrm{Conf}_n(X)$ the quotient of $\mathrm{PConf}_n(X)$ by the action of Σ_n in the category of smooth schemes. We denote by $\widetilde{\mathrm{Conf}}_n(X)$ the stacky quotient $[\mathrm{PConf}_n(X)/\Sigma_n]$. Observe that in this case the Σ_n-action is free which implies that the assumptions of the previous proposition are satisfied.

3. Motives

3.1. **Generalities.** We recall the construction of the category $\mathbf{DA}(S,\Lambda)$. We start from the category of complexes of presheaves of Λ-modules on the site of smooth schemes over S and we force descent for étale hypercovers, contractibility of the affine line and invertibility of the Tate motive for the tensor product. We define $\mathbf{DM}(S,\Lambda)$ in a similar fashion except that we start from the category of complexes of presheaves with transfers and we use the Nisnevich topology instead of the étale topology. There is a left adjoint functor

$$\mathbf{DM}(S,\Lambda) \to \mathbf{DA}(S,\Lambda)$$

which is an equivalence of categories when Λ is a $\mathbb{Q}$-algebra. The category $\mathbf{DM}(S,\Lambda)$ contains a collection of objects indexed by $\mathbb{Z}$ called the Tate twists and denoted by $\Lambda(n), n \in \mathbb{Z}$. There are analogously defined motives in $\mathbf{DA}(S,\Lambda)$. There is a symmetric monoidal category structure on both $\mathbf{DM}(S,\Lambda)$ and $\mathbf{DA}(S,\Lambda)$ such that the formula

$$\Lambda(i) \otimes \Lambda(j) \cong \Lambda(i+j)$$

holds for all integers i and j.

A smooth scheme over S yields an object in $\mathbf{DM}(S,\Lambda)$ and in $\mathbf{DA}(S,\Lambda)$ denoted $M(X)$ and $M_{et}(X)$ respectively. The étale motivic cohomology with coefficients in Λ of a smooth scheme X over S is the bigraded collection of Λ-modules :

$$\mathrm{H}^{p,q}_{et}(X,\Lambda) := \mathrm{Hom}_{\mathbf{DA}(S,\Lambda)}(M_{et}(X), \Lambda(q))$$

Likewise the motivic cohomology is given by

$$\mathrm{H}^{p,q}(X,\Lambda(q)) := \mathrm{Hom}_{\mathbf{DM}(S,\Lambda)}(M(X), \Lambda)$$

We can also define the motivic cohomology of $[X/G]$ when G is a finite group acting on a smooth scheme X. For this we first define $M([X/G])$ as the homotopy colimit of the simplicial object of $\mathbf{DM}(S,\Lambda)$ given by

$$[n] \mapsto M(X \times G^n)$$

and then we set

$$\mathrm{H}^{p,q}([X/G],\Lambda) := \mathrm{Hom}_{\mathbf{DM}(S,\Lambda)}([X/G], \Lambda(q)).$$

Remark 3.1. The object $M([X/G])$ can equivalently be defined as the homotopy orbits of the G-action on $M(X)$.

Remark 3.2. The careful reader will have noticed that there is an abuse of terminology in the definition above as there is no definition of the homotopy colimit of a simplicial object in a triangulated category. What we really mean is that $\mathbf{DM}(S,\Lambda)$ is the homotopy category of a model category (or an ∞-category) and that the diagram $[n] \mapsto M(X \times G^n)$ lifts to the level of model categories. We can thus take the homotopy colimit in the model category and then consider the result as an object in the homotopy category. We will allow ourselves to make this abuse in a few other places in the paper.

We could define in a smilar fashion the étale motive of $[X/G]$. We have an isomorphism

$$M_{et}([X/G]) \cong M(X/G)$$

whenever X/G exists by Proposition 2.1. Since étale motivic cohomology coincides with motivic cohomology when the ring of coefficients is a $\mathbb{Q}$-algebra, we can also deduce the following theorem.

Theorem 3.3. *Assume that the quotient X/G exists in the category of smooth schemes. Let Λ be a $\mathbb{Q}$-algebra. Then the canonical map $[X/G] \to X/G$ induces an isomorphism*

$$\mathrm{H}^{p,q}(X/G,\Lambda) \to \mathrm{H}^{p,q}([X/G],\Lambda).$$

Another important feature of these categories that we now recall is the so-called purity theorem.

Theorem 3.4. *Let X be a smooth scheme and D be a smooth closed subscheme of X of codimension c, then there exists a cofiber sequence in $\mathbf{DM}(S,\Lambda)$.*

$$M(X-D) \xrightarrow{M(i)} M(X) \to M(D)(d)[2d]$$

where i denotes the open inclusion $X - D \to X$. There is a similar sequence in $\mathbf{DA}(S,\Lambda)$.

3.2. **Betti realization.** The functor $X \mapsto X_{an}$ from smooth S-schemes to topological spaces induces a functor called Betti realization

$$B^* : \mathbf{DA}(S,\Lambda) \to \mathbf{D}(\Lambda)$$

and similarly a functor

$$B^* : \mathbf{DM}(S,\Lambda) \to \mathbf{D}(\Lambda)$$

The only things we will need to know about these functors is that they are symmetric monoidal left adjoints (in fact they come from left Quillen functors) and:

Fact 3.5. *The composite*

$$B^* \circ M : \mathrm{Sm}_S \to \mathbf{D}(\Lambda)$$

is naturally isomorphic to the functor $X \mapsto C_*(X_{an},\Lambda)$, where Sm_S denotes the category of smooth schemes over S.

3.3. **Mixed Tate motives.** We denote by $\mathbf{DAT}(S,\Lambda)$ the smallest triagulated subcategory of $\mathbf{DA}(S,\Lambda)$ containing all the Tate twists $\Lambda(n)$ and closed under arbitrary coproducts and retracts. We define $\mathbf{DMT}(S,\Lambda)$ analogously. An object of $\mathbf{DAT}(S,\Lambda)$ or $\mathbf{DMT}(S,\Lambda)$ will be called a mixed Tate motive. Observe that the tensor product on $\mathbf{DM}(S,\Lambda)$ and $\mathbf{DA}(S,\Lambda)$ induces a symmetric monoidal structure on $\mathbf{DMT}(S,\Lambda)$ and $\mathbf{DAT}(S,\Lambda)$ respectively.

Proposition 3.6. *Let X be a smooth scheme over S that is such that $M(X)$ is a mixed Tate motive, then for each n, $M(\mathrm{PConf}_n(X))$ is also a mixed Tate motive.*

Proof. Let $P(n)$ be the set of subsets of $\{1,\dots,n\}$ with 2 elements. For P a subset of $P(n)$, we denote by $\mathrm{PConf}_{n,P}(X)$ the complement in X^n of the diagonals indexed by P. We have $\mathrm{PConf}_{n,\varnothing}(X) = X^n$ and $\mathrm{PConf}_{n,P(n)}(X) = \mathrm{PConf}_n(X)$. We shall prove more generally that $M(\mathrm{PConf}_{n,P}(X))$ is mixed Tate for all P. We do this by induction on the pair $(n,|P|)$ with the lexicographic ordering. The result is obvious for $|P|=0$. Now, assume that (i,j) is an element of P and let $Q = P - \{(i,j)\}$. Then we can decompose $\mathrm{PConf}_{n,Q}(X)$ as the union of the open subscheme $\mathrm{PConf}_{n,P}(X)$ and its closed complement which is isomorphic to $\mathrm{PConf}_{n-1,P'}(X)$ for a certain $P' \in P(n-1)$. The result thus follows from the induction hypothesis and the purity theorem. □

3.4. **Betti realization and connectivity.** Finally, we will need the following theorem which relates the connectivity of a mixed Tate motive to the connectivity of its Betti realization. We denote by $\mathbf{DAT}(S,\Lambda)_{gm}$ the smallest thick subcategory of $\mathbf{DAT}(S,\Lambda)$ that contains all the Tate twists $\Lambda(i), i \in \mathbb{Z}$.

Theorem 3.7. *Let $f\colon X \to Y$ be a map in $\mathbf{DAT}(S,\mathbf{Z})_{gm}$. Assume that $B(f)$ is an isomorphism in negative homological degrees. Then, for all commutative rings Λ, the map*

$$\mathrm{Hom}_{\mathbf{DA}(S,\Lambda)}(Y,\Lambda(q)[p]) \to \mathrm{Hom}_{\mathbf{DA}(S,\Lambda)}(X,\Lambda(q)[p])$$

induced by f is an isomorphism for all q and for all $p<0$.

Proof. Using the extension of scalars adjunction

$$\mathbf{DA}(S,\mathbb{Z}) \leftrightarrows \mathbf{DA}(S,\Lambda)$$

we see that it suffices to prove that the map induced by f

$$\mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(Y,\Lambda(q)[p]) \to \mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(X,\Lambda(q)[p])$$

is an isomorphism for all q and for all $p<0$. We will in fact prove the more general fact that the map

$$\mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(Y,A(q)[p]) \to \mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(X,A(q)[p])$$

is an isomorphism for any abelian group A. The case $A=\mathbf{Q}$ is classical and follows from the existence of the motivic t-structure on $\mathbf{DAT}(S,\mathbf{Q})_{gm}$ [Lev93] and the argument of [Hor16, Corollary 4.6] (beware that, contrary to what is stated in [Hor16], this corollary is incorrect with integral coefficients. However, it is correct with rational coefficients).

We will prove the result for $A=\mathbf{Z}/n$ in the paragraph below. Assuming this for the moment, then we can conclude as follows. Since the functor $A \mapsto \mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(U,A(q)[p])$ preserves filtered colimits, it is enough to prove the statement for A finitely generated. Since this functor also preserves direct sums, we can reduce to $A=\mathbf{Z}/n$ or $A=\mathbf{Z}$. By assumption, we have the result for $A=\mathbf{Z}/n$. Now, for any $U \in \mathbf{DAT}(S,\mathbf{Z})_{gm}$, we have a long exact sequence

$$\dots \to \mathrm{Hom}(U,\mathbf{Q}/\mathbf{Z}(q)[p-1]) \to \mathrm{Hom}(U,\mathbf{Z}(q)[p]) \to \mathrm{Hom}(U,\mathbf{Q}(q)[p]) \to \dots$$

(where all Homs are in $\mathbf{DA}(S,\mathbf{Z})$) induced by the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

By the five lemma, the result for $\mathbf{Z}$ will follow from the result for $\mathbf{Q}$ and the result for $\mathbf{Q}/\mathbf{Z}$. To conclude in the latter case, it suffices to observe that $\mathbf{Q}/\mathbf{Z}$ is a filtered colimit of cyclic groups.

Now, we treat the case $A=\mathbf{Z}/n$. Suslin rigidity gives us an equivalence $\mathbf{DA}(S,\mathbf{Z}/n) \simeq \mathbf{D}(S_{et},\mathbf{Z}/n)$. Moreover, viewed through the equivalence, the Betti realization functor can be identified with

$$i^* : \mathbf{D}(S_{et},\mathbf{Z}/n) \to \mathbf{D}(T_{et},\mathbf{Z}/n) \simeq \mathbf{D}(\mathbf{Z}/n)$$

where $T_{et} = \mathrm{Spec}(\overline{K})$ is the small étale site of the algebraic closure of K and i^* is the map induced by the inclusion $i\colon K \to \overline{K}$ (this is the Suslin rigidity theorem together with Artin's theorem on the isomorphism between étale and singular cohomology). Moreover, for $U \in \mathbf{D}(S_{et}, \mathbf{Z}/n)$, we have an étale descent spectral sequence of the form

$$H^s(\Gamma, \mathrm{Hom}_{\mathbf{D}(T_{et},\mathbf{Z}/n)}(i^*U, \mathbf{Z}/n(q)[t])) \implies \mathrm{Hom}_{\mathbf{D}(S_{et},\mathbf{Z}/n)}(U, \mathbf{Z}/n(q)[s+t])$$

where $H^*(\Gamma, -)$ denotes Galois cohomology with respect to $\Gamma = \mathrm{Gal}(\overline{K}/K)$ (see for instance [AGV72, VIII, Corollaire 2.3]). Coming back to our situation, the map $f\colon X \to Y$ will induce an isomorphism on the E^2-page of the étale descent spectral sequence in the range $t < 0$. Since moreover, this spectral sequence is zero for negative s, we see that the map $f\colon X \to Y$ will induce an isomorphism on the E^∞-page for $s + t < 0$ which implies that we indeed have an isomorphism

$$\mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(Y, \mathbf{Z}/n(q)[p]) \to \mathrm{Hom}_{\mathbf{DA}(S,\mathbf{Z})}(Y, \mathbf{Z}/n(q)[p])$$

for $p < 0$. □

4. Construction of the stabilization map

In this section X is a smooth scheme over S.

Assumption 4.1. We assume that there exists a pair (Y, D) consisting of Y a smooth scheme over S, D a non-empty closed smooth subscheme such that $X \cong Y - D$. We make the additional assumption that D has a K-point.

The first part of the assumption is not very restrictive. Such a pair can be found as soon as X is not complete. Indeed, in that case, by a theorem of Nagata, X can be written as the complement of a non-empty closed subscheme in a complete scheme $\overline{X}$. Then, using Hironaka's resolution of singularities, we can assume that $\overline{X}$ is smooth and that $\Delta = \overline{X} - X$ is a normal crossing divisor. If we write $\Delta = \cup_{i=0}^n D_i$ the decomposition of Δ into irreducible components, we can then take $Y = \overline{X} - \cup_{i=1}^n D_i$ and $D = D_0 \cap Y$.

Under Assumption 4.1, we construct (see Construction 4.5 below) a stabilization map

$$M(\mathrm{PConf}_n(X)) \to M(\mathrm{PConf}_{n+1}(X)) \tag{4.1}$$

that is Σ_n-equivariant (see Lemma 4.6) in the category $\mathbf{DM}(S, \Lambda)$. We can thus take homotopy orbits with respect to the symmetric group and get a map

$$M(\widetilde{\mathrm{Conf}}_n(X)) \to M([\mathrm{PConf}_{n+1}(X)/\Sigma_n])$$

Finally we can compose this with the obvious map

$$M([\mathrm{PConf}_{n+1}(X)/\Sigma_n]) \to M([\mathrm{PConf}_{n+1}(X)/\Sigma_{n+1}]) \cong M(\widetilde{\mathrm{Conf}}_{n+1}(X))$$

and we get the stabilization map

$$M(\widetilde{\mathrm{Conf}}_n(X)) \to M(\widetilde{\mathrm{Conf}}_{n+1}(X)). \tag{4.2}$$

This is the map that will induce the isomorphisms in Theorems 5.1 and 6.7. The maps (4.1) and (4.2) have the correct Betti realizations by Lemma 4.7. Observe that in the étale setting, by Proposition 2.1, the stabilization map above can equally be seen as a map of étale motives:

$$M(\mathrm{Conf}_n(X)) \to M(\mathrm{Conf}_{n+1}(X))$$

Proposition 4.2. *Let X be a smooth scheme and D be a smooth closed subscheme of codimension c, let $N(j)$ be the normal bundle of the inclusion $j : D \to X$ and let $N_0(j)$ be the complement of the zero section of $N(j)$. Then there exists a map called the motivic exponential map*

$$\exp^0 : M(N_0(j)) \to M(X - D)$$

whose Betti realization is homotopic to the map induced by the tubular neighborhood inclusion

$$C_*(N_0(j)_{an}) \subset C_*((X - D)_{an})$$

Proof. The closed inclusions j and $D \subset N(j)$ induce distinguished (Gysin) triangles

$$M(X - D) \to M(X) \to M(D)(c)[2c] \quad \text{and} \quad M(N_0(j)) \to M(N(j)) \to M(D)(c)[2c].$$

In [Lev07, Section 5.2] Levine shows that one can construct an exponential map

$$\exp^0 : M(N_0(j)) \to M(X - D)$$

such that the following square in $\mathbf{DM}(S, \Lambda)$ commutes and is homotopy cocartesian:

$$\begin{tikzcd} M(N_0(j)) \arrow[r] \arrow[d] & M(N(j)) \arrow[d] \\ M(X - D) \arrow[r] & M(X) \end{tikzcd} \tag{4.3}$$

where the right-hand vertical map is induced by the composition $N(j) \to D \xrightarrow{j} X$. Note that Levine construct $\exp^0$ in the stable motivic homotopy category $\mathbf{SH}(S)$ instead of $\mathbf{DM}(S, \Lambda)$ but the latter is simply the category of modules over the motivic Eilenberg-MacLane spectrum in $\mathbf{SH}(S)$ so we can simply tensor Levine's map with the motivic Eilenberg-MacLane spectrum to get the desired map.

The map $\exp^0$ has the correct Betti realization. Indeed, since the Betti realization functor can be modelled by a left adjoint ∞-functor between stable ∞-categories, it preserves homotopy cartesian squares. Therefore, applying Betti realization to the square 4.3, we get a homotopy cartesian square in $\mathbf{D}(\Lambda)$:

$$\begin{tikzcd} C_*(N_0(j)_{an}) \arrow[r] \arrow[d] & C_*(N(j)_{an}) \arrow[d] \\ C_*((X - D)_{an}) \arrow[r] & C_*(X_{an}) \end{tikzcd} \tag{4.4}$$

On the other hand, we know from the excision theorem in classical algebraic topology and the tubular neighborhood theorem that there is a homotopy cocartesian (and hence also cartesian) square in $\mathbf{D}(\Lambda)$ which is the same as the one above but where the left-hand vertical map is induced by the tubular neighborhood inclusion. Since homotopy pullbacks are unique up to weak equivalences, the Betti realization of $\exp^0$ must the homotopic to the map induced by the tubular neighborhood inclusion. □

Proposition 4.3. *In the setting of Proposition 4.2, suppose that X is equipped with an action of a discrete group G that sends D to itself. Equip the normal bundle $N(j)$ with the natural induced G-action, which restricts to a G-action on the subscheme $N_0(j)$. Then the motivic exponential map map of Proposition 4.2 is G-equivariant.*

Proof. An examination of the construction of Levine (see [Lev07, Section 5.2]) shows that the map $\exp^0$ is natural in the data (X, D). □

Definition 4.4. Denote by $\mathrm{PConf}_{n+1,\leqslant 1}(Y, D) \subset \mathrm{PConf}_{n+1}(Y)$ the smooth subscheme of ordered $(n+1)$-point configurations in Y where the first n points lie in X, and denote by $\mathrm{PConf}_{n+1,1}(Y, D)$ its closed smooth subscheme given by those configurations where, in addition, the last point lies in D. Note that $\mathrm{PConf}_{n+1,1}(Y, D)$ is isomorphic to $\mathrm{PConf}_n(X) \times D$ and its complement $\mathrm{PConf}_{n+1,\leqslant 1}(Y, D) - \mathrm{PConf}_{n+1,1}(Y, D)$ is isomorphic to $\mathrm{PConf}_{n+1}(X)$.

Construction 4.5. The stabilization map (4.1) is constructed as follows. Denote the inclusion $D \to Y$ by i and the inclusion $\mathrm{PConf}_{n+1,1}(Y, D) \to \mathrm{PConf}_{n+1,\leqslant 1}(Y, D)$ by j. Choose a K-point $* \in N_0(i)$ (such a point exists by our assumption that D has a K-point). By construction, we have an identification $N_0(j) = \mathrm{PConf}_n(X) \times N_0(i)$, so the choice of $*$ induces a map

$$M(\mathrm{PConf}_n(X)) \to M(N_0(j)). \tag{4.5}$$

By Proposition 4.2 and the identification of $\mathrm{PConf}_{n+1,\leqslant 1}(Y, D) - \mathrm{PConf}_{n+1,1}(Y, D)$ with $\mathrm{PConf}_{n+1}(X)$, there is a map

$$M(N_0(j)) \to M(\mathrm{PConf}_{n+1}(X)), \tag{4.6}$$

and (4.1) is defined to be the composition of these two maps.

Lemma 4.6. *The stabilization map* (4.1) *is* Σ_n*-equivariant.*

Proof. There is an action of Σ_n on $\mathrm{PConf}_{n+1,\leqslant 1}(Y, D)$ given by permuting the first n points, and this action preserves its subscheme $\mathrm{PConf}_{n+1,1}(Y, D)$. This induces an action on $N_0(j)$, which corresponds, under the identification $N_0(j) = \mathrm{PConf}_n(X) \times N_0(i)$, to the permutation action on $\mathrm{PConf}_n(X)$ and the trivial action on $N_0(i)$. Hence the map (4.5) is Σ_n-equivariant. By Proposition 4.3, the map (4.6) is also Σ_n-equivariant. □

Lemma 4.7. *The Betti realizations of* (4.1) *and of* (4.2) *are the maps of chain complexes induced by the classical stabilization maps for ordered and unordered configuration spaces respectively.*

Proof. The classical stabilization map for ordered configuration spaces is homotopic to the composition of

$$x \mapsto (x, *)\colon \mathrm{PConf}_n(X)_{an} \to \mathrm{PConf}_n(X)_{an} \times N_0(i)_{an} \cong N_0(j)_{an} \tag{4.7}$$

with the tubular neighborhood inclusion

$$N_0(j)_{an} \to \mathrm{PConf}_{n+1}(X)_{an}. \tag{4.8}$$

We therefore need to show that $C_*((4.7); \Lambda) \cong B^*((4.5))$ and $C_*((4.8); \Lambda) \cong B^*((4.6))$. The first of these identifications follows directly from Fact 3.5 and the second follows from the last part of Proposition 4.2. This shows that the Betti realization of (4.1) is the map induced by the classical stabilization map for ordered configuration spaces.

The classical stabilization map $s^{un}\colon \mathrm{Conf}_n(X)_{an} \to \mathrm{Conf}_{n+1}(X)_{an}$ for unordered configuration spaces is obtained from the classical stabilization map $s^{ord}\colon \mathrm{PConf}_n(X)_{an} \to \mathrm{PConf}_{n+1}(X)_{an}$ for ordered configuration spaces by taking quotients with respect to the natural symmetric group actions. Since these actions are free and properly discontinuous, we may equivalently take homotopy orbits instead of quotients. At the beginning of this section, we defined (4.2) by taking homotopy orbits (in the category of motives) of symmetric group actions on the map (4.1). The fact that $C_*(s^{un}; \Lambda) \cong B^*((4.2))$ thus follows from the fact that $C_*(s^{ord}; \Lambda) \cong B^*((4.1))$ – proved in the paragraph above – and the fact that both B^* and $C_*(-; \Lambda)$ preserve homotopy colimits, since they are both left adjoints. □

5. Homological stability for open schemes

In this section, we still assume that X is a smooth S-scheme that can be written as $X = Y - D$ with Y a smooth scheme and D a non-empty smooth closed subscheme of Y. We further assume that we have chosen an S-point of the punctured normal bundle of the inclusion $D \to Y$. Hence, we have a stabilization map

$$M(\widetilde{\mathrm{Conf}}_n(X)) \to M(\widetilde{\mathrm{Conf}}_{n+1}(X))$$

as explained in the previous section. This stabilization map induces a map in étale motivic cohomology

$$\mathrm{H}^{p,q}_{et}(\mathrm{Conf}_{n+1}(X),\Lambda) \cong \mathrm{H}^{p,q}_{et}(\widetilde{\mathrm{Conf}}_{n+1}(X),\Lambda) \to \mathrm{H}^{p,q}_{et}(\widetilde{\mathrm{Conf}}_n(X),\Lambda) \cong \mathrm{H}^{p,q}_{et}(\mathrm{Conf}_n(X),\Lambda).$$

Theorem 5.1. *Assum that X satisfies the condtions above and assume further that $M(X)$ is a mixed Tate motive and that X_{an} is a connected topological space. Then the stabilization map*

$$\mathrm{H}^{p,q}_{et}(\mathrm{Conf}_{n+1}(X),\Lambda) \to \mathrm{H}^{p,q}_{et}(\mathrm{Conf}_n(X),\Lambda)$$

is an isomorphism for $p \leq n/2$.

Proof. This follows from the fact that the Betti realization detects connectivity (Theorem 3.7) and the statement in the topological case (Theorem 1.1). □

6. Stability for $X = \mathbf{A}^d$

In the case where $X = \mathbf{A}^d$ we can prove stability in the category $\mathbf{DM}(S,\Lambda)$ instead of $\mathbf{DA}(S,\Lambda)$. Note that a similar result was claimed in [Hor16] but the proof is incorrect. It was based on the claim that the Betti realization functor on $\mathbf{DMT}(S,\Lambda)$ detects connectivity but this is not the case. Here we replace the Betti realization by the associated graded for the weight filtration which indeed detects connectivity and is still sufficiently close to the Betti realization in this case.

6.1. **The weight filtration.** For $X \in \mathbf{DMT}(S,\Lambda)$, we denote by

$$\ldots \to w_{\geq n}(X) \to w_{\geq n-1}(X) \to \ldots \to X$$

the weight filtration of X. If we denote by $w_{\geq n}\mathbf{DMT}(\Lambda)$ the smallest localizing subcategory of $\mathbf{DMT}(\Lambda)$ containing the objects $\Lambda(i)$ with $i \geq n$, then $w_{\geq n}$ is by definition the right adjoint to the inclusion

$$w_{\geq n}\mathbf{DMT}(\Lambda) \to \mathbf{DMT}(\Lambda)$$

We denote by $w_n(X)$ the cofiber of the map $w_{\geq n+1}(X) \to w_{\geq n}(X)$. We will need the following fact about this functor.

Proposition 6.1. *Let X be an object of $\mathbf{DMT}(S,\Lambda)$. Then the motive $w_n(X)$ is of the form $A(n)$ for some object A of $\mathbf{D}(\Lambda)$.*

Proposition 6.2. *Let $M \in \mathbf{DMT}(\Lambda)$. Assume that for each n, the object $B \circ w_n(M)$ is connective, then for $m < 0$, we have*

$$\mathrm{H}^{m,q}(M,\Lambda) = 0$$

Proof. The filtration

$$\ldots \to w_{\geq n}(M) \to w_{\geq n-1}(M) \to \ldots \to M$$

induces a spectral sequence

$$E_1^{u,v} = \mathrm{H}^{u+v,q}(w_u(M),\Lambda) \implies \mathrm{H}^{u+v,q}(M,\Lambda)$$

therefore, it suffices to prove that $\mathrm{H}^{u+v,q}(w_uM,\Lambda)=0$ for $u+v<0$. By the previous Proposition, the object $w_u(M)$ is of the form $A(u)$ where A is in $\mathbf{D}(\Lambda)$. The fact that $B\circ w_u(M)$ is connective implies that A is connective. It follows that A lies is in the full subcategory of $\mathbf{D}(\Lambda)$ generated under colimits by $\Lambda[0]$. Hence, without loss of generality, we may assume that $A=\Lambda[0]$. But then the result is true since the group $\mathrm{H}^{n,q}(S,\Lambda)$ vanishes whenever $n<0$. □

6.2. **The weight filtration of** $\mathrm{PConf}_n(\mathbf{A}^d)$**.** We will need to compute the weight filtration of the scheme $\mathrm{PConf}_n(\mathbf{A}^d)$. For this purpose, we introduce a definition. We take α a positive rational number and we write $\alpha=p/q$ with p and q two positive coprime integers.

Definition 6.3. We say that an object $X\in\mathbf{DM}(S,\Lambda)$ is α-pure if the following two conditions are satisfied.

- If n is not a multiple of q, then the map
$$B(w_{\geq n+1}(X))\to B(w_{\geq n}(X))$$
is an isomorphism.
- If n is a multiple of q then the map
$$B(w_{\geq n}(X))\to B(X)$$
exhibits $B(w_{\geq n}(X))$ as the αn-connective cover of $B(X)$.

In other words, an object is α-pure if, up to rescaling by α, the weight filtration induces the Postnikov filtration upon application of the Betti realization functor. Observe that if $q\neq 1$, then, $B(X)$ has homology concentrated in degrees that are multiple of p.

Example 6.4. Take $X=M(\mathbf{P}^n)$. Then it is a classical computation that
$$X=\Lambda(0)\oplus\Lambda(1)[2]\oplus\ldots\oplus\Lambda(n)[2n],$$
and it follows that X is 2-pure.

For an example where α is not an integer, take $X=M(\mathbf{A}^d-\{0\})$. Then one has
$$X=\Lambda(0)\oplus\Lambda(d)[2d-1]$$
and we easily see that X is $(\frac{2d-1}{d})$-pure.

Proposition 6.5. *Write $\alpha=p/q$ with p and q two coprime positive integers.*

1. *If X is α-pure, then $B(w_{qm}(X))\cong\mathrm{H}_{pm}(X_{an})[pm]$ and $B(w_n(X))=0$ if n is not a multiple of q.*
2. *If X is α-pure, then $X(p)[q]$ is also α-pure.*
3. *If X and Z are α-pure and $Y\in\mathbf{DM}(\Lambda)$ fits in a cofiber sequence*
$$X\to Y\to Z$$
then Y is also α-pure.

Proof. We prove (1). If n is not a multiple of q, then, by definition, the map
$$B(w_{\geq n+1}(X))\to B(w_{\geq n}(X))$$
is an isomorphism, this implies that $w_n(X)=0$. If $n=qm$, we apply B to the cofiber sequence
$$w_{\geq qm+1}(X)\to w_{\geq qm}(X)\to w_{qm}(X).$$
By definition, $B(w_{\geq qm}(X))$ is the pm-connective cover of $B(X)$ and $B(w_{\geq qm+1}(X))$ is the $p(m+1)$-connective cover of X, it follows that $B(w_m(X))\cong H_{pm}(X_{an})[pm]$.

The proof of (2) is elementary, once we observe that $w_{\geq n}(X(p))\cong w_{\geq n-p}(X)$.

In order to prove (3), we use an alternative characterization of α-pure objects. An object X of $\mathbf{DM}(\Lambda)$ is α-pure if the homology of $B(w_{\geq n}(X))$ concentrated in degrees $\geq \lceil \alpha n \rceil$ and the homology of $B(w_{<n}(X))$ is concentrated in degrees $< \lceil \alpha n \rceil$. $\square$

Lemma 6.6. *The object $w_k(\mathrm{PConf}_n(\mathbf{A}^d))$ is trivial if d does not divide k and*

$$w_{kd}(\mathrm{PConf}_n(\mathbf{A}^d)) \cong \mathrm{H}_{k(2d-1)}(\mathrm{PConf}_n(\mathbf{C}^d))[k(2d-1)].$$

Proof. By (1) of the previous proposition, it suffices to prove that the motive of $\mathrm{PConf}_n(\mathbf{A}^d)$ is $(\frac{2d-1}{d})$-pure. We will prove more generally that this result holds for the complement of a good arrangement of codimension d subspaces of an affine space. So let us consider $X = \mathbb{A}^n - \bigcup_{i\in I} V_i$ a complement of a good arrangement of codimension d subspaces. We proceed by induction on the cardinality of I. The result is obvious if I is empty. Now assume that $I = J \sqcup \{i\}$. Let us denote by $Y = \mathbb{A}^n - \bigcup_{j\in J} V_j$. Then, X is an open subset of Y whose closed complement, denoted Z, is the complement of a good arrangement of codimension d subspaces. We thus have a cofiber sequence

$$M(Z)(d)[2d-1] \to M(X) \to M(Y)$$

thus the result follows from the induction hypothesis and (2) and (3) of Proposition 6.5. $\square$

6.3. The main theorem.

Theorem 6.7. *The map*

$$\mathrm{H}^{p,q}(\mathrm{Conf}_{n+1}(\mathbf{A}^d), \Lambda) \to \mathrm{H}^{p,q}(\mathrm{Conf}_n(\mathbf{A}^d), \Lambda)$$

is an isomorphism for $p \leq n/2$.

Proof. By Proposition 6.2, it suffices to prove that, for each a, the stabilization map

$$w_a(\mathrm{Conf}_n(\mathbf{A}^d)) \to w_a(\mathrm{Conf}_{n+1}(\mathbf{A}^d))$$

is $n/2$-connected. Since w_a commutes with colimits, it is equivalent to prove that

$$w_a(\mathrm{PConf}_n(\mathbf{A}^d))_{\Sigma_n} \to w_a(\mathrm{PConf}_{n+1}(\mathbf{A}^d))_{\Sigma_{n+1}}$$

is $n/2$-connected. By Lemma 6.6, the domain and codomain are both zero if d does not divide a (so the map is ∞-connected), so we may assume that $a = kd$.

By Lemma 6.6 and the spectral sequence $\mathrm{H}_*(G; \mathrm{H}_*(C)) \Rightarrow \mathrm{H}_*(C_G)$ for a G-equivariant chain complex C, we have

$$\mathrm{H}_l(w_{kd}(\mathrm{PConf}_n(\mathbf{A}^d))_{\Sigma_n}) \cong \mathrm{H}_{l-k(2d-1)}(\Sigma_n, \mathrm{H}_{k(2d-1)}(\mathrm{PConf}_n(\mathbf{C}^d))),$$

so it suffices to prove that the map

$$\mathrm{H}_{l-k(2d-1)}(\Sigma_n, \mathrm{H}_{k(2d-1)}(\mathrm{PConf}_n(\mathbf{C}^d))) \to \mathrm{H}_{l-k(2d-1)}(\Sigma_{n+1}, \mathrm{H}_{k(2d-1)}(\mathrm{PConf}_{n+1}(\mathbf{C}^d)))$$

is an isomorphism in the range $l \leq n/2$. By Lemma 6.8 below, this map is an isomorphism in the range

$$l \leq n/2 + k(2d-1) - k.$$

Note that $k(2d-1) - k \geq 0$, since $k \geq 0$ and $d \geq 1$, so we are done. $\square$

Lemma 6.8. *For any dimension $m \geq 2$ and coefficient ring Λ, the stabilization map*

$$\mathrm{H}_q(\Sigma_n, \mathrm{H}_r(\mathrm{PConf}_n(\mathbf{R}^m); \Lambda)) \to \mathrm{H}_q(\Sigma_{n+1}, \mathrm{H}_r(\mathrm{PConf}_{n+1}(\mathbf{R}^m); \Lambda))$$

is an isomorphism for $q \leq \frac{n}{2} - \frac{r}{m-1}$.

Proof. For fixed $m \geq 2$, the assignment $S \mapsto \mathrm{PConf}_S(\mathbf{R}^m)$, where S is a finite set, naturally extends to a functor $\mathrm{FI}\sharp \to \mathrm{hTop}$, by [CEF15, Proposition 6.4.2], and hence the assignment

$$S \mapsto \mathrm{H}_r(\mathrm{PConf}_S(\mathbf{R}^m); \Lambda)$$

extends to a functor $\mathrm{FI}\sharp \to \Lambda\text{-Mod}$. We will show that this functor is *polynomial of degree* $\leq 2r/(m-1)$. This will imply the stated result, by [Pal18, Theorem A] with $M = \mathbf{R}^\infty$ and $X = *$. (Theorem A of [Pal18] is stated only in the case $\Lambda = \mathbf{Z}$, so that Λ-Mod is the category of abelian groups, but the results of [Pal18] generalise immediately to any abelian category, including Λ-Mod.)

As an aside, we note that applying [RWW17, Theorem A] with $\mathcal{C} = \mathrm{FI}$, $A = \varnothing$, $X = \{*\}$ (where we may take $k = 2$) and $N = 0$ also gives the stated result, although only in the worse range of degrees $q \leq \frac{n}{2} - \frac{2r}{m-1} - 1$. The stated result would also follow from [Bet02, Theorem 4.3] if the functor $\mathrm{H}_r(\mathrm{PConf}_\bullet(\mathbf{R}^m); \Lambda)\colon \mathrm{FI}\sharp \to \Lambda\text{-Mod}$ could be extended further to the category of finite sets and all partially-defined functions (not just partially-defined injections). However, there does not seem to be a natural extension to a functor on this larger category.

We now show that the functor $T = \mathrm{H}_r(\mathrm{PConf}_\bullet(\mathbf{R}^m); \Lambda)\colon \mathrm{FI}\sharp \to \Lambda\text{-Mod}$ is polynomial of degree at most $d = \lfloor 2r/(m-1) \rfloor$. Recall that this means that $\Delta^{d+1}T = 0$, where Δ is the operation on functors from $\mathrm{FI}\sharp$ to an abelian category defined in [Pal18, §3.1]. In the terminology of [Dja16] this is equivalent to saying that T is *strongly polynomial of strong degree at most* d, when considered as a functor on the subcategory $\Theta = \mathrm{FI} \subset \mathrm{FI}\sharp$. (Note that the operation Δ of [Pal18] corresponds to the operation δ_1 of [Dja16] and, in the case $\mathcal{M} = \Theta$ of [Dja16], it suffices to consider only δ_1, rather than δ_a for all objects a of $\mathcal{M}$, since Θ is generated as a monoidal category by the object 1.) Thus [Dja16, Proposition 4.4] implies that it is equivalent to prove that $T|_{\mathrm{FI}}$ is *generated in degrees at most* d. This, in turn, is equivalent, by [CEF15, Remark 2.3.8], to the condition that $H_0(T|_{\mathrm{FI}})_i = 0$ for all $i > d$, where H_0 is the left adjoint of the inclusion of $\mathrm{Fun}(\mathrm{FB}, \Lambda\text{-Mod})$ into $\mathrm{Fun}(\mathrm{FI}, \Lambda\text{-Mod})$, where FB is the category of finite sets and bijections. The proof of Theorem 4.1.7 of [CEF15] shows that this condition will hold as long as, for all $n \geq 0$, the Λ-module

$$T(n) = \mathrm{H}_r(\mathrm{PConf}_n(\mathbf{R}^m); \Lambda)$$

is generated by at most $O(n^d)$ elements. It therefore remains to verify this last condition.

We first consider the case $\Lambda = \mathbf{Z}$. By [FH01, Theorem V.4.1], the $\mathbf{N}$-graded ring

$$\mathrm{H}^*(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$$

is generated by $\binom{n}{2}$ elements, all in degree $m-1$, subject to certain relations (the cohomological Yang-Baxter relations). It follows that the abelian group $\mathrm{H}^r(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$ is trivial unless $r = i(m-1)$, in which case it is generated by the set of (commutative) monomials of degree i in $\binom{n}{2}$ variables. The number of such monomials is at most

$$\binom{n}{2}^i \sim O(n^{2i}) = O(n^d),$$

and so $\mathrm{H}^r(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$ is generated by at most $O(n^d)$ elements. Since the cohomology groups $\mathrm{H}^*(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$ are free in all degrees, by [FH01, Theorem V.1.1], and the space $\mathrm{PConf}_n(\mathbf{R}^m)$ has the homotopy type of a finite CW-complex (see [FH01, §VI.8–10]), the universal coefficient theorem implies that the homology groups $\mathrm{H}_*(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$ are also free in all degrees and $\mathrm{H}_r(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$ has the same rank as $\mathrm{H}^r(\mathrm{PConf}_n(\mathbf{R}^m); \mathbf{Z})$, so it is also generated by at most $O(n^d)$ elements.

Finally, going back to the case of an arbitrary ring Λ, the universal coefficient theorem implies that there is an isomorphism of Λ-modules

$$\mathrm{H}_r(\mathrm{PConf}_n(\mathbf{R}^m);\Lambda) \cong \mathrm{H}_r(\mathrm{PConf}_n(\mathbf{R}^m);\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda,$$

so $\mathrm{H}_r(\mathrm{PConf}_n(\mathbf{R}^m);\Lambda)$ is generated as a Λ-module by at most $O(n^d)$ elements. $\square$

Remark 6.9. The slightly weaker version of Lemma 6.8 with the smaller range of degrees $q \leq \frac{n}{2} - \frac{2r}{m-1} - 1$, which would follow from applying [RWW17, Theorem A] instead of [Pal18, Theorem A] (see the second paragraph of the proof above) would suffice for the proof of the main theorem (Theorem 6.7), *except* in dimension $d = 1$.

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