GT : Six functors formalisms

Talk 3

Introduction

- Germinated in Grothendieck's mind between 1956 and 1963 during the development of étale cohomology of schemes, and it was soon realized that very similar formalisms also exist in other contexts, such as cohomology of topological spaces, or for D-modules on algebraic varieties in char 0.
- This machinery subsumes and captures different behaviors concerning cohomology theories such as analogies of Poincaré duality, base change formulas, and more.

I. Around Poincaré duality.

↳ the formalism of six operations notably allows to formulate a "Poincaré duality" type thm for smooth objects" in a given context.
Let us recall the most classical Poincaré duality thm and its generalization to arbitrary topological manifolds.

Theorem : (Poincaré duality on compact oriented manifolds)

X compact oriented topological manifold of dim d.
A be an abelian group
we have an iso of graded abelian groups

$$H^{d-*}(X,A) \cong H_*(X,A)$$

coming from $C^*(X,A)[d] \simeq C_*(X,A)$ in $\mathcal{D}(\mathbb{Z})$, the derived ∞ - category of abelian groups.

Remark : $\mathcal{D}(\mathbb{Z})$ is a stable ∞ - category which can be understood as the one presented by the model category of chain cx of abelian groups with the projective model structure.

Definition : Let X be a locally simply path connected topological space.

A local system on X is a functor

$$\underset{\substack{\uparrow \\ \text{fundamental} \\ \text{groupoïd}}}{\Pi_1(X)} \longrightarrow \underset{\substack{\nwarrow \\ \text{category} \\ \text{of abelian groups}}}{Ab}$$

Remark : • This definition is equivalent to locally constant sheaf of abelian groups.

• local system = a system of coefficients for twisted cohomology.

Exemple : X a topological manifold

$\tilde{X}$ the associated two-sheeted orientation cover

By Galois correspondance, it corresponds to a functor

$$\Pi_1(X) \longrightarrow \{\{a,b\}\}$$

$\nwarrow$ category avec un unique objet $\{a,b\}$

Sending this set to the abelian group $\mathbb{Z}$ by $\{a,b\} \longrightarrow \langle a,b \mid ab\rangle$, we get a local system

$$\tilde{\mathbb{Z}} : \Pi_1(X) \longrightarrow Ab.$$

If $A : \Pi_1(X) \longrightarrow Ab$ is any local system on X, we define $\tilde{A}$ to be $A \otimes \tilde{\mathbb{Z}}$.

$\rightsquigarrow$ A manifold is orientable if and only if $\tilde{\mathbb{Z}}$ is the constant sheaf $\mathbb{Z}$.

We will now define appropriate analogues of chains and cochains with coefficients in a local system.

Let X be a topological manifold and $A : \Pi_1(X) \longrightarrow Ab$ a local system.
Let b_n be the barycenter of Δ^n, $n \geq 0$ and γ_i be the line between b_n and the barycenter of the i-th face.

Singular chains with coefficients in a local system

Let $n \geq 0$
$$C_n(X,A) = \left\{ \sum_{\sigma : \Delta^n \to X} a_\sigma \sigma \;\middle|\; a_\sigma \in A(\sigma(b_n)) \text{ and only finitely many terms are } \neq 0 \right\}$$

$$\partial(a_\sigma \sigma) = \sum_{i=0}^{m} (-1)^i A(\gamma_i)(a_\sigma) d_i^* \sigma .$$

Singular cochains with coefficients in a local system

Let $m \geq 0$ $\quad C^m(X, A) = \left\{ \varphi : \mathrm{Hom}_{\mathrm{Top}}(\Delta^m, X) \to \coprod_{x \in X} A(x) \;\middle|\; \begin{array}{l} \varphi(\sigma) \in A(\sigma(b_m)) \\ \forall \sigma : \Delta^m \to X \end{array} \right\}$

The codifferential is defined dually as in the case above.

Remark: For A = trivial local coefficient system (cst value $A \in Ab$ + paths are sent to id_A) we get the usual singular chains & cochains.

Borel - Moore singular chains

$$C_*(X, A) \subset C_*^{BM}(X, A)$$

$$C_m^{BM}(X, A) = \left\{ \sum_{\sigma : \Delta^m \to X} a_\sigma \sigma \;\middle|\; \begin{array}{l} a_\sigma \in A(\sigma(b_m)) \text{ and } \forall K \subset X \text{ compact} \\ \{\sigma \mid a_\sigma \neq 0 \ \ \sigma(\Delta^m) \cap K \neq \emptyset\} \text{ is finite} \end{array} \right\}$$

Compactly supported cochains

$$C_c^*(X, A) \subset C^*(X, A)$$

$$C_c^m(X, A) = \left\{ \varphi \in C^m(X, A) \mid \exists K \subset X \text{ compact with } \varphi_{X \setminus K} = 0 \right\}$$

Theorem: (Twisted Poincaré duality)

we have isomorphisms in $\mathcal{D}(\mathbb{Z})$:

$$C_c^*(X, \tilde{A})[d] \xrightarrow{\sim} C_*(X, A)$$

and

$$C^*(X, \tilde{A})[d] \xrightarrow{\sim} C_*^{BM}(X, A)$$

II. Interlude on sheaves with values in an ∞-category $\mathcal{C}$

Definition: Let X be a topological space. A sieve on X is a set $\mathcal{U}$ of open subsets of X such that if $V \in \mathcal{U}$ and $U \subset V$ then $U \in \mathcal{U}$.

If $W = \bigcup_{V \in \mathcal{U}} V$, we say that the sieve $\mathcal{U}$ covers W.

Let $\mathcal{F} \in PSh(X, \mathcal{C})$ be a presheaf with values in an ∞-cat $\mathcal{C}$, i.e a functor $\mathcal{F} : Open(X)^{op} \to \mathcal{C}$. We say that $\mathcal{F}$ is a sheaf if for all sieves $\mathcal{U}$ on X covering $W \in Open(X)$, we have

$$\mathcal{F}(W) \xrightarrow{\sim} \lim_{V \in \mathcal{U}^{op}} \mathcal{F}(V).$$

We denote by $Sh(X, \mathcal{C})$ the ∞-category of sheaves with values in $\mathcal{C}$.

Exemple : $Sh(X, \mathcal{D}(\mathbb{Z}))$

III. Grothendieck key idea

$\rightsquigarrow$ The fundamental object of a cohomology theory $H^*(-,-)$ is some kind of functorial association

$$\begin{array}{ccc} \{\text{geometric context}\} & \longrightarrow & \infty\text{-Cat} \\ X & \longmapsto & \{\text{Natural coefficients for } H^*(X,-)\} \end{array}$$

↳ the fonctoriality of this association should be one of the features that a six-functor formalism encodes.

Examples of coefficients :

- <u>For ordinary cohomology of abelian groups</u>

$$\begin{array}{ccc} \{\text{Top spaces}\} & \longrightarrow & \infty\text{-Cat} \\ X & \longmapsto & Sh(X, \mathcal{D}(\mathbb{Z})) \end{array}$$

- <u>For generalized cohomology theories</u>

$$\begin{array}{ccc} \{\text{Top spaces}\} & \longrightarrow & \infty\text{-Cat} \\ X & \longmapsto & Sh(X, Sp) \end{array}$$

where Sp stands for the stable ∞-cat of spectra.

- <u>For quasi-coherent cohomology</u>

$$\begin{array}{ccc} \{\text{schemes}\} & \longrightarrow & \infty\text{-Cat} \\ X & \longmapsto & QCoh(X) \end{array}$$

↑ ∞-category of quasi-coherent sheaves.

[See, Lecture VIII]

- <u>For de Rham cohomology</u>

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{quasi-compact} \\ \text{quasi-separated} \\ \text{schemes of } \mathbb{K} \\ \text{of char } 0 \end{array} \right\} & \longrightarrow & \infty\text{-Cat} \\ X & \longmapsto & Crys(X) \end{array}$$

↑ category of $\mathcal{D}$-modules.

[See, Appendix to Lecture VIII]

IV. Some topological definitions

Definition: X, Y, Z be topological spaces and $f : X \to Y$ a continuous map. We say that:

- f is universally closed if $\forall\, Z \to Y$ the pullback map $f : X' \to Z$ is closed.
- f is separated if the diagonal map $\Delta : X \to X \times_Y X$ is univ. closed.
- f is proper if f is universally closed and separated.
- f is locally proper if $\forall\, x \in X$ and $\forall\, U \in \mathcal{U}_x$, $\exists\, A \subset U$, $A \in \mathcal{U}_x$ and $B \in \mathcal{U}_{f(x)}$ such that $f : A \to B$ is a proper map.
- f is smooth if $\forall\, x \in X$ there is an open neighbourhood U of x and V of $f(x)$ such that $f(U) \subset V$ and

$$\begin{tikzcd} U \arrow[r, "\sim"] \arrow[dr, "f"'] & V \times D \arrow[d, "pr_1"] \\ & V \end{tikzcd} \qquad \text{for } D \subset \mathbb{R}^d,\ d \geq 0$$

- f is étale if it is a local homeomorphism (smooth of relative dim zero).

Definition: Let X be a topological space.
We say that:

- X is quasi-compact if $X \to *$ is universally closed.
- X is Hausdorff if $X \to *$ is separated.
- X is proper if $X \to *$ is proper.
- X is locally proper if $X \to *$ is locally proper.
- X is a manifold if $X \to *$ is smooth.

V. Six-functor formalism for topological spaces

↳ we study appropriate functorialities of $X \to Sh(X, Sp)$, in order to give an intro to the six-functor formalism.
A formal development is presented in next section.

The three following adjoint functors form the soul of a six-functor formalism:

Pullback and pushforward

A map $f : X \to Y$ always gives

$$f^* : Sh(Y, Sp) \underset{\perp}{\rightleftarrows} Sh(X, Sp) : f_*$$

where $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ and f^* is the unique left adjoint.

Exceptional direct / inverse image

If f is locally proper, we have in addition

$$f_! : Sh(X, Sp) \underset{\perp}{\rightleftarrows} Sh(Y, Sp) \; f^!$$

If f is separated, we can define

$$f_! \mathcal{F}(V) = \varinjlim_{\substack{K \subset f^{-1}(V) \\ K \to V \\ \text{proper}}} \mathcal{F}_K(f^{-1}(V))$$

where if $Z \subset X$ is closed, we define $\mathcal{F}_Z(U)$ as the homotopy fiber of $\mathcal{F}(U) \to \mathcal{F}(U \cap (X \setminus Z))$.
If f is not separated, we can proceed by descent using locally separatedness.

$Sh(X, Sp)$ is a symmetric monoidal category.

It gives an adjunction $\otimes \dashv Hom$.

These functors also have the following compatibilities

- Base change formula ⇝ compatibility between $*$ and $!$

  For every $f: X \to Y$ locally proper and every cartesian square

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

  by the adjunction

  we have a natural iso $g_Y^* f_! \simeq f'_! g_X^*$ $\quad (\Leftrightarrow f^! g_{Y,*} \simeq g_{X,*} f'^!)$

- The functor f^* is symmetric monoidal ⇝ compatibility between $*$ and $\otimes$

- Projection formula ⇝ compatibility between $!$ and $\otimes$

  The functor $f_!$ is $Sh(Y, Sp)$-linear, i.e $f_!(A \otimes f^* B) \simeq f_!(A) \otimes B$

- If f is locally proper and separated, there is a natural map

$$f_! \to f_*$$

  which is an iso if f is proper.

- If $j: U \to X$ is étale (ex: an open embedding), $j_!$ is the extension by zero functor, and therefore left adjoint to j^*.

- For $f: X \to *$, X a locally contractible space and $A: \pi_1(X) \to Ab$ a local system, the sheaf cohomology

$$f_* A = \Gamma(X, A) \in \mathcal{D}(\mathbb{Z})_{\leq 0}$$

  coincides with $C^*(X, A)$.

Definition: Let $Pic(Sh(X, Sp))$ be the full subcategory of invertible objects wrto the monoidal structure.

Exemple: $\mathbb{Z}[d]$ for $d \in \mathbb{Z}$
$\tilde{\mathbb{Z}}$ are invertible.

⇝ tensoring by such objects are interpreted as shifts and twists.

Definition : (Co) homologies in a six - functor formalism

Let $E \in Sp$ and $\xi \in Pic(Sh(X, Sp))$.

Let X be a topological space and $f : X \to *$.

- The cohomology of X with coefficients in E twisted by ξ

$$H^*(X, E, \xi) = \Gamma(X, E, \xi) = f_*(\xi \otimes f^* E).$$

If X is locally proper,

- The homology of X with coefficients in E twisted by ξ

$$H_*(X, E, \xi) = f_!(\xi^{-1} \otimes f^! E)$$

- The cohomology with compact support of X with coefficients in E twisted by ξ

$$H^*_c(X, E, \xi) = \Gamma_c(X, E, \xi) = f_!(\xi \otimes f^* E)$$

- The Borel - Moore homology of X with coefficients in E twisted by ξ

$$H^{BM}_*(X, E, \xi) = f_*(\xi^{-1} \otimes f^! E)$$

Remark : For A a local system on X seen as a locally cst sheaf of abelian groups and therefore as object in the heart of $Sh(X, D(\mathbb{Z}))$. we can define :

- $H^*(X, A) = f_*(A \otimes f^* \mathbb{Z}) = f_* A$
- $H_*(X, A) = f_!(A \otimes f^! \mathbb{Z})$
- $H^*_c(X, A) = f_!(A \otimes f^* \mathbb{Z}) = f_! A$
- $H^{BM}_*(X, A) = f_*(A \otimes f^! \mathbb{Z})$

- Künneth formula

Le X be locally proper and Y be an arbitrary top space.

Let us consider

$$\begin{tikzcd} X \times Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "p"] & X \arrow[d, "p_X"] \\ Y \arrow[r, "p_Y"'] & * \end{tikzcd}$$

We have an isomorphism :

$$- \boxtimes - : p_1^*(-) \otimes p_2^*(-) : Sh(X, Sp) \otimes Sh(Y, Sp) \to Sh(X \times Y, Sp).$$

Remark : If $f: X \to Y$ is proper, the Projection formula becomes

$$f_* A \otimes B \to f_*(A \otimes f^* B),$$

which is adjoint to $f^*(f_* A \otimes B) \cong f^* f_* A \otimes f^* B \to A \otimes f^* B$

- Poincaré duality.

Definition : Let $f: X \to Y$ be a locally proper map of topological spaces. The sheaf $\omega_f = f^!(\mathbb{Z})$ is called the dualizing sheaf of f. For $f: X \to *$, we write ω_X.

Remarque : If X is a manifold of dim d, then $\omega_X \otimes \mathbb{Z} \cong \tilde{\mathbb{Z}}[d]$

Theorem : Let $f: X \to Y$ be a smooth map of relative dim d. The natural map $\omega_f \otimes f^* \to f^!$ is an isomorphism.

Remarque : Twisted Poincaré Duality follows from the above isom :

$$H_c^*(X, \tilde{A})[d] = f_!(\tilde{A} \otimes f^* \mathbb{Z}[d])$$

$$H_*(X, A) = f_!(A \otimes \tilde{\mathbb{Z}}[d])$$

- Verdier duality

Theorem : Let $f: X \to Y$ be a proper and smooth map. Then the functor f_* admits a right adjoint given by $f^* \otimes \omega_f$.

VI. Six-functor formalisms.

Let $\mathcal{C}$ be an ∞-category with all finite products.
Let E be a class of morphisms stable under pullback and composition (and containing all iso).

Definition : The symmetric monoidal ∞-category of correspondances $\mathrm{Corr}(C, E)$ is given as follows.

1. The objects are objects of C.
2. The symmetric monoidal structure is the Cartesian symmetric monoidal structure of C.

3. The morphisms are correspondances: $\mathrm{Hom}_{\mathrm{Corr}(C,E)}(X,Y)$ is given by the (∞)-groupoid of objects $W \in C$ together with maps

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \qquad \text{where } g \in E .$$

4. The composition of morphisms is given by the composition of correspondances, i.e. the composite of $X \leftarrow W \rightarrow Y$ and $Y \leftarrow W' \rightarrow Z$ is given by

$$\begin{tikzcd} & & W \times_Y W' \arrow[dl] \arrow[dr] & & \\ & W \arrow[dl] \arrow[dr] & & W' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd} .$$

Remark: The above definition will be made more precise next talk.

Definition: • A 3-functor formalism is a lax symmetric monoidal functor
$$D : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}_\infty .$$

• A 6-functor formalism is a 3-func. f. for which $-\otimes A$, f^* and $f_!$ admit right adjoint.

Remark: This encodes 3-functors: $\otimes$ f^*, $f_!$ and all their relations:

1. On objects, D defines $X \mapsto D(X)$.
2. The lax sym. mono. defines a natural "exterior tensor product"
$$D(X) \otimes D(X) \to D(X \times X) .$$
with the diagonal pullback bellow, this defines the tensor product $\otimes$ on $D(X)$.
3. For any map $f : X \to Y$, the correspondance $Y \xleftarrow{f} X = X$ defines the pullback functor $f^* : D(Y) \to D(X)$.
4. For any map $f : X \to Y$ in E, the correspondance $X = X \xrightarrow{f} Y$ defines the functor $f_! : D(X) \to D(Y)$.

A correspondance $X \xleftarrow{f} W \xrightarrow{g} Y$ gets sent to $g_! f^* : D(X) \to D(Y)$
The compatibility of this with composition amount to base change formula.