LECTURE §III§: SYMM. MONOIDAL ∞CATEGORIES and 3 FUNCTOR FORMALISM

§0. MOTIVATION

Recall: WANT a "6-FUNCTOR FORMALISM" for C ∞category w-x-
$E \subseteq \mathrm{Morph}(C)$ stable under -∘-
pullback
isomorphs
"EXCEPTIONAL MORPHISM"

Main Example: • C = loc. cmpct Hausdorff top spaces
E = "compactificable maps"
i.e. maps which write as

$$f = \bar{f} \circ j$$

$\bar{f}$: PROPER MAPS
j: OPEN IMMERSION

• C = QCoherent schemes
E = separated morphisms of finite type

<u>where</u> A 6 FUNCTOR FORMALISM should be:

EASY to FORMALIZE (it is a functor $C^{op} \longrightarrow$ cat of s.m. ∞-cat) $\rightsquigarrow$

1) $C \ni X \longmapsto D(X) \in Cat_\infty$
2) $(D(X), \otimes)$ s.m. structure
3) $\forall f: X \longrightarrow Y$, a PULLBACK functor $f^*: D(Y) \longrightarrow D(X)$
compatible w/ $- \otimes -$, $(\ )^*$ functorial

4) $\forall f: A \longrightarrow B$ in E, an EXCEPTIONAL pushforward $f_!: D(X) \longrightarrow D(Y)$
$(\ )_!$ functorial on E, BASE CHANGE, PROJECTION FORMULA

This is the HARD PART!

5) RIGHT ADJOINTS: $- \otimes - \dashv \underline{Hom}(-,-)$
$f^* \dashv f_*$
$f_! \dashv f^!$

$\rightsquigarrow$ **SOLUTION** (to have $\otimes, f^*, f_!$) : **Define a sym. mon. cat $Corr(C,E)^{\otimes}$ so that (1)—(4) $\Longleftrightarrow$ LAX S.M. FUNCTOR $Corr(C,E)^{\otimes} \longrightarrow Cat_\infty$**

§1. The ∞-category $\mathrm{Corr}(C,E)$

<u>DEF</u> (last time) C ∞-cat w/ finite limits
$E \subseteq \mathrm{Morph}(C)$ stable under $-\circ-$, pullback, $\ni$ isos

$\mathrm{Corr}(C,E)$ should be the ∞-category with:

- $\mathrm{Ob}(\mathrm{Corr}(C,E)) = \mathrm{ob}(C)$
- morphisms: $X \xrightarrow{\phi} Y \iff$
$$\begin{tikzcd} & W \arrow[dl] \arrow[dr, "\in E"] & \\ X & & Y \end{tikzcd}$$
- composition:
$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & & L \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd} \iff \begin{tikzcd} & & W\times_Y L \arrow[dl] \arrow[dr, "\in E"] \arrow[dll, bend right] \arrow[drr, bend left] & & \\ X & W \arrow[l] \arrow[r, "E?"] & Y & L \arrow[l] \arrow[r] & Z \end{tikzcd}$$

Also would like a s.m. structure

<u>where</u> $X \otimes Y = X \times Y$

NOW: $\to$ $\mathrm{Corr}(C,E)$ as an ∞cat.
$\to$ What are s.m. ∞categories and $\mathrm{Corr}(C,E)^{\otimes}$ s.m. struct

DEF • $\forall n$, $(\Delta^n)^2_+ \subseteq (\Delta^n)^{op} \times \Delta^n$ subsimpl set spanned by (i,j) w/ $i \geq j$

$\underline{n=1}$

$$\begin{tikzcd}
 & (1,0) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1)
\end{tikzcd}$$

$\underline{n=2}$

$$\begin{tikzcd}
 & & (2,0) \arrow[dl] \arrow[dr] & & \\
 & (1,0) \arrow[dl] \arrow[dr] & & (2,1) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2)
\end{tikzcd}$$

$\rightsquigarrow$ $(\Delta^\bullet)^2_+ : \Delta \longrightarrow sSets$ (COSIMPLICIAL ∞CATEGORY)

• $Corr(C,E)_\bullet \subseteq Fun^\infty((\Delta^\bullet)^2_+, C)$ subsimplicial set where

$f \in Corr(C,E)_n \iff \begin{cases} \text{ALL ARROWS DOWN RIGHT} \in E \\ \text{ALL SMALL SQUARES are CARTESIAN} \end{cases}$

$\underline{n=1}$

$$\begin{tikzcd}
 & X_{10} \arrow[dl] \arrow[dr, "\in E"] & \\
X_{00} & & X_{11}
\end{tikzcd}$$

$\underline{n=2}$

$$\begin{tikzcd}
 & & X_{20} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\checkmark"] & & \\
 & X_{10} \arrow[dl] \arrow[dr] & & X_{21} \arrow[dl] \arrow[dr] & \\
X_{00} & & X_{11} & & X_{22}
\end{tikzcd}$$

$x_{ij} \in ob(C)$

<u>Rmk</u> Well defined bc $f \in \mathrm{Corr}(C,E)_n \Rightarrow d_i f \in \mathrm{Corr}(C,E)_{n-1} \ \forall i$
$s_j f \in \mathrm{Corr}(C,E)_{n+1} \ \forall j$

ex

$$d_1\left(\begin{tikzcd}[column sep=small] & & X_{20} \arrow[dl] \arrow[dr] & & \\ & X_{10} \arrow[dl] \arrow[dr] & & X_{21} \arrow[dl] \arrow[dr] & \\ X_{00} & & X_{11} & & X_{22} \end{tikzcd} \right) = \begin{tikzcd}[column sep=small] & X_{20} \arrow[dl] \arrow[dr, "\in E"] & \\ X_{00} & & X_{22} \end{tikzcd} \qquad s_0(X) = \begin{tikzcd}[column sep=small] & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & & X \end{tikzcd} \quad \text{etc..}$$

$$d_0\left(\qquad \right) = \begin{tikzcd}[column sep=small] & X_{21} \arrow[dl] \arrow[dr] & \\ X_{11} & & X_{22} \end{tikzcd}$$

<u>PROP</u> $\mathrm{Corr}(C,E)_\bullet$ is an ∞category, i.e. HORN FILLING CONDITION

i.e. $\forall n \geq 2$, $\forall 0<i<n$

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall f"] \arrow[d, hook] & \mathrm{Corr}(C,E)_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists \bar{f}"'] & \end{tikzcd}$$

<u>PROOF</u> [Lemma 6.1.2 of [LZ12a]]

For $n=2$, $i=1$

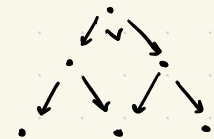
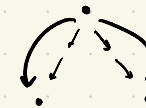
$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "f"] \arrow[d, hook] & \mathrm{Corr}(C,E)_\bullet \\ \Delta^2 \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

$$f \rightsquigarrow \begin{tikzcd}[column sep=small] & & X_{10} \times_{X_{11}} X_{21} \arrow[dl] \arrow[d] \arrow[dr] & & \\ & X_{10} \arrow[dl] \arrow[dr] & & X_{21} \arrow[dl] \arrow[dr] & \\ X_{00} & & X_{11} & & X_{22} \end{tikzcd}$$

$\bar{f}$ can be obtained via

$\boxtimes$

Rmk In particular, we know that composition in the ∞-cat $\mathrm{Corr}(C;E)$ corresponds to what we wanted

(**Recall**: Take  $\overset{\text{FILLING COND}}{\rightsquigarrow}$  $\overset{\text{take } d_1}{\rightsquigarrow}$ )

$\leftrightsquigarrow \quad \Lambda^2_1 \longrightarrow \mathrm{Corr}(C;E) \qquad\qquad \leftrightsquigarrow \quad \Delta^2 \longrightarrow \mathrm{Corr}(C;E)$

§2. SYMM. MONOIDAL ∞-CATEGORIES

Goal | way to be able to define $\bigotimes_{i\in I} X_i \quad \forall I$ finite set, $X_i \in ob(C)$ |

CLASSICAL DEF of s.m. 1-cat (w/ $-\otimes - : C\times C \to C$
$1 : * \to C$
isos: $x\otimes(y\otimes z) \xrightarrow{\sim} (x\otimes y)\otimes z$
$x\otimes y \xrightarrow{\sim} y\otimes x$
$1\otimes x \xrightarrow{\sim} x$
+ coherence [penthagon + exhagon axioms]) $\rightsquigarrow$ HARD to GENERALIZE (HIGHER COHERENCES!)

$\rightsquigarrow$ Reformulation: "A S.M. 1-cat = COMMUTATIVE MONOID in $(\mathrm{Cat}, \times)$"

What is a COMMUTATIVE MONOID in Cat_∞ ? ∞cat of ∞categories

DEF Fin. category w/ objects FINITE POINTED SETS, denoted by $\langle I\rangle = I \cup \{*\}$, I finite set
morphisms ⇝ maps of finite pointed sets

Special objs $\langle n\rangle = \{1, \dots, n\} \cup \{*\}$

Some special maps
- $\forall I$ fin set, $\forall i \in I$ $\quad \langle I\rangle \xrightarrow{\rho_i} \langle i\rangle$, $\quad j \longmapsto \begin{cases} i & \text{if } j = i \\ * & \text{otherwise} \end{cases}$ "projections"
- $\beta : \langle 2\rangle \longmapsto \langle 1\rangle$ "multiplication"
  $* \longrightarrow *$
  $1 \longrightarrow 1$
  $2 \longrightarrow 1$

DEF A S.M. ∞CATEGORY is a functor $X : \mathrm{Fin}_* \longrightarrow \mathrm{Cat}_\infty$ s.t. $\forall I$ fin set
the map $X(\langle I\rangle) \xrightarrow{\sim} \prod_{i \in I} X(\langle i\rangle)$ induced by $\langle I\rangle \xrightarrow{\rho_i} \langle i\rangle$
is an equivalence.
("Segal condition")

<u>Good def bc</u> • objectwise, X determined by $X(\langle 1\rangle) \in \mathrm{Cat}_\infty$ $\qquad \mathcal{D} = X(\langle 1 \rangle)$

• TENSOR PRODUCT : should be $-\otimes- : \mathcal{D}\times\mathcal{D} \longrightarrow \mathcal{D}$

$\quad$ ↳ it is the image of $\beta : \langle 2\rangle \longrightarrow \langle 1\rangle$!!

<u>Indeed</u> $X(\beta) : X(\langle 2\rangle) \underset{\text{Sp}}{\simeq} X(\langle 1\rangle)\times X(\langle 1\rangle) = \mathcal{D}\times\mathcal{D} \longrightarrow X(\langle 1\rangle) = \mathcal{D}$

and it is : • SYMMETRIC <u>because</u>

$$\begin{tikzcd} \langle 2\rangle \arrow[rr, "(12)"] \arrow[dr, "\beta"'] & & \langle 2\rangle \arrow[dl, "\beta"] \\ & \langle 1\rangle & \end{tikzcd} \implies \begin{tikzcd} \mathcal{D}\times\mathcal{D} \arrow[r, "-\otimes-"] \arrow[d, "(12)"] & \mathcal{D} \\ \mathcal{D}\times\mathcal{D} \arrow[ur, "-\otimes-"'] & \end{tikzcd}$$

meaning $a\otimes b \simeq b\otimes a$ (ALL HIGHER coherences encoded)

• ASSOCIATIVE <u>because</u>

$$\begin{tikzcd} \langle 3\rangle \arrow[r] & \langle 2\rangle \arrow[r, "\beta"] & \langle 1\rangle \end{tikzcd} \quad \underset{\text{in Fin.}}{\equiv} \quad \begin{tikzcd} \langle 3\rangle \arrow[r] & \langle 2\rangle \arrow[r, "\beta"] & \langle 1\rangle \end{tikzcd}$$

$$\begin{array}{ccc} 1 \to 1 & & 1 \\ 2 \searrow & 1 \to 1 & \\ 3 \to 2 & 2 \nearrow & \end{array} \qquad \begin{array}{ccc} 1 \searrow & 1 \to 1 & \\ 2 \to 1 & 2 \nearrow & \\ 3 \to 2 & & \end{array}$$

$$\implies \begin{tikzcd} \mathcal{D}^{\times 3} \arrow[r, "(\mathrm{id}{,}-\otimes-)"] \arrow[d, "(\otimes{,}\mathrm{id})"'] & \mathcal{D}^{\times 2} \arrow[d, "\otimes"] \\ \mathcal{D}^2 \arrow[r, "\otimes"'] & \mathcal{D} \end{tikzcd}$$

• UNITAL <u>because</u> $X(\langle \emptyset\rangle) \simeq * \longrightarrow X(\langle 1\rangle) = \mathcal{D}$

while it is MORE COMPLICATED if $\mathcal{C}$ has another s.m struct ...

<u>Rmk</u> If $\mathcal{C}$ is ∞cat w/ finite prods, a commutative monoid in $\mathcal{C}$ is $\mathrm{Fin}_* \longrightarrow \mathcal{C}$ w/ Segal condition

PRACTICAL PROBLEM: Hard to write functors into Cat_∞

Already at 1-categorical level, where usually one has PSEUDOFUNCTORS i.e. $X(f\circ g) \overset{\simeq}{\underset{\text{only iso}}{\longrightarrow}} X(f)\circ X(g)$

no longer a pbm in Cat_∞ but there higher coherences

Pbm in 1Cat solved by GROTHENDIECK

<u>Thm</u> There is an EQUIVALENCE of 2-CATEGORIES

$$\mathrm{PseudoFunctors}(C, Cat) \simeq \mathrm{coFib}(C)$$

COCARTESIAN FIBRATIONS over C

<u>where</u> • $p: E \to C$ is coCartesian fibration if $\forall\, p(x) \xrightarrow{f} c$ morph in C

$\exists$ lift $x \xrightarrow{\bar f} \bar c$ $(p(\bar f) = f)$ which is INITIAL amongst lifts } "COCARTESIAN LIFT"

i.e.

$$\begin{tikzcd} x \arrow[rr, bend left, "h"] \arrow[dr, "\bar f"'] & & \bar d \\ & \bar c \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$$p(x) \xrightarrow{f} c \longrightarrow d$$

• The correspondence goes as:

$$\begin{array}{ccc} \mathrm{PsFun}(C, Cat) & \longleftarrow & \mathrm{coFib}(C) \\ F: C \to Cat & \longleftarrow & (E \xrightarrow{p} C) \\ c \mapsto p^{-1}(c) =: E_c & & \end{array}$$

$$\begin{array}{ccc} \mathrm{PsFun}(C, Cat) & \longrightarrow & \mathrm{coFib}(C) \\ (F: C \to Cat) & \longmapsto & \left(\int_F^C \xrightarrow{p} C\right) \end{array}$$

where $\int_F C$ is a category whose: • Objs $= (c \in C,\ x \in Ob(F(c)))$

• Morphisms $(c,x) \longrightarrow (d,y)$ is $\begin{cases} f: c \to d \text{ in } C \\ \begin{array}{ccc} Fc & \xrightarrow{Ff} & Fd \ni y \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ x & \longmapsto & Ff(x) \end{array} \text{ and } \phi: Ff(x) \to y \end{cases}$

$\int_F C \xrightarrow{p} C$ forgetful functor.

There is an ANALOGOUS RESULT for ∞CATEGORIES

<u>DEF</u> $p: E \to C$ b/ ∞cats is a <u>CoCartesian fibration</u> if $\begin{cases} p \text{ inner fibration} \\ \forall\, p(x) \xrightarrow{f} c \quad \exists\ x \xrightarrow{\bar{f}} \bar{c} \text{ lift which is INITIAL} \\ \quad (\text{i.e. space of } \dashrightarrow \text{ is contractible}) \\ \quad \text{"locally coCartesian lift"} \\ \text{locally coCart. lifts stable under composition} \end{cases}$

<u>Rmk</u> $p: E \to C$ coCart fib of 1-cat

$\iff N(p): N(E) \longrightarrow N(C)$ coCart fib of ∞cat

Thm [Straightening-Unstraightening, Lurie] There is an EQUIVALENCE of $(\infty,1)$-categories

$$\mathrm{Fun}^\infty(C, \mathrm{Cat}_\infty) \simeq \mathrm{coCartFib}(C)$$

where on objects it works as the 1-categ case

Rmk It would be nat equiv of $(\infty,2)$-cat. But we take $\mathrm{Fun}^\infty(C, \mathrm{Cat}_\infty)$ ONLY INVERTIBLE NATURAL TRANSFORM

$\Rightarrow$ in the ∞cat of coCartFib we restrict to
$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & C & \end{tikzcd}$$
st F preserves loc. coCart. lifts

DEF" A S.M. ∞CAT is a COCARTESIAN FIBRATION $\mathcal{C}^{\otimes} \xrightarrow{p} \mathrm{Fin}_\bullet$ s.t. $\forall I \in$ fin set, the nat map $(\mathcal{C}^{\otimes})_{\langle I\rangle} \xrightarrow{\sim} \prod_{i\in I} (\mathcal{C}^{\otimes})_{\langle i\rangle}$ induced by coCart lifts of $\langle I\rangle \xrightarrow{e_i} \langle i\rangle$ is an equiv of ∞categories

We call $(\mathcal{C}^{\otimes})_{\langle 1\rangle} =: \mathcal{C}$ the UNDERLYING ∞CATEGORY of p.

$\underset{\sim\sim}{\text{In particular}}$ $X : \mathrm{Fin}_* \longrightarrow \mathrm{Cat}_\infty$ s.m 1-cat , $C = X(\langle 1\rangle)$

$\underset{\text{straight-unst}}{\leftrightsquigarrow}$ $p : C^{\otimes} \longrightarrow \mathrm{Fin}_*$. $\quad Ob(C^{\otimes}) = \{ (c_i)_{i\in I} \mid c_i \in C,\ I \in \text{fin sets} \}$

$$C^{\otimes}\big((c_i)_{i\in I}, (d_j)_{j\in J}\big) = \begin{cases} \alpha : \langle I \rangle \longrightarrow \langle J \rangle \text{ in } \mathrm{Fin}_* \text{ and} \\ \forall j \in J \quad f_j : \bigotimes_{i\in\alpha^{-1}(j)} c_i \longrightarrow d_j \end{cases}$$

$-\otimes-$ is the coCartesian lift of $\beta : \langle 2\rangle \longrightarrow \langle 1\rangle$

<u>Rmk</u> Def$^{\text{n}}$ of s.m ∞cat is apparently <u>not</u> self dual, but via def' and $\underset{C \longrightarrow C^{op}}{\mathrm{Cat}_\infty \simeq \mathrm{Cat}_\infty}$ it is.

<u>SPECIAL CASE</u>

[2.4.1.4, 2.4.1.8 HA]

C w/ finite prods $\Rightarrow$ C^{op} w/ finite coprods
and $\underset{\text{(ess. unique)}}{\exists!}$ $(C^{op})^{\sqcup} \longrightarrow \mathrm{Fin}_*$ sym. monoidal category $\underset{\sim\sim}{\text{where}}$ $-\otimes- = -\sqcup-$

C w/ finite prods $\Rightarrow$ $\underset{\text{(ess. unique)}}{\exists!}$ $C^{\times} \longrightarrow \mathrm{Fin}_*$ s.m. structure on C $\underset{\sim\sim}{\text{where}}$ $-\otimes- = -\times-$

$\underset{\sim\sim}{\text{In particular}}$ $C = \mathrm{Cat}_\infty$

FINAL NOTION NEEDED : LAX S.M. FUNCTOR

1-Categorical: $\phi : (C, \otimes) \longrightarrow (D, \otimes)$ w/ natural maps $\bigotimes_{i \in I} \phi(X_i) \longrightarrow \phi(\bigotimes_{i \in I} X_i)$ $\forall I$

$\rightsquigarrow$ want existence of natural transformations

$$\begin{tikzcd} C^{\times I} \arrow[r, "\phi^{\times I}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow"] & D^{\times I} \arrow[d, "\otimes"] \\ C \arrow[r, "\phi"'] & D \end{tikzcd}$$

$\rightsquigarrow$ $F_C : \mathrm{Fin}_* \to \mathrm{Cat}$
$F_D : \mathrm{Fin}_* \to \mathrm{Cat}$

$$\begin{tikzcd} F_C(I) \arrow[r, "\phi_I"] \arrow[d, phantom, "\cong"] & F_D(I) \arrow[d, phantom, "\cong"] \\ \prod_i F_C(\langle i \rangle) \arrow[r, "\prod_i \phi_i"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow"] & \prod_i F_D(\langle i \rangle) \arrow[d, "\otimes"] \\ F_C(\langle 1 \rangle) \arrow[r, "\phi_{\langle 1 \rangle}"'] & F_D(\langle 1 \rangle) \end{tikzcd}$$

i.e.
- pseudonat transformation (not necess invertible)
- $\phi_I \simeq \prod_{i \in I} \phi_i$

DEF (LAX S.M. FUNCTOR b/ ∞CATS)

$\overset{\text{STRAIGHT}}{\underset{\text{-UNSTRAIGHT}}{\longleftrightarrow}}$

$$\begin{tikzcd} C^{\otimes} \arrow[rr, "\tilde{\phi}"] \arrow[dr, "p"'] & & D^{\otimes} \arrow[dl, "p'"] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

$\tilde{\phi} : C^{\otimes} \longrightarrow D^{\otimes}$ over Fin_* st $\tilde{\phi}$ preserves loc cocart lifts of $\rho_i : \langle I \rangle \longrightarrow \langle i \rangle$

(then ϕ s.m $\longleftrightarrow$ nat transf are invertible $\leftrightarrow$ $\tilde{\phi}$ preserve all loc cocart lifts)

§3. S.M. STRUCTURE on $\mathrm{Corr}(C, E)$ for (C, E) geometric context

Goal: Define $\mathrm{Corr}(C,E)^{\otimes} \longrightarrow \mathrm{Fin}_*$ coCartesian fibration such that

$$\begin{cases} \mathrm{Corr}(C,E)^{\otimes}_{\langle 1 \rangle} \simeq \mathrm{Corr}(C,E) \\ \text{s.m. structure so that } C^{op} \to \mathrm{Corr}(C,E) \text{ is symm. monoidal} \end{cases}$$

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow f & & {\scriptstyle f}\swarrow \quad \searrow \!\! = \\ y & & y \qquad\quad y \end{array}$$

(so that in $\mathrm{Corr}(C,E)$) $X \otimes Y = X \times Y$

DEF $\mathrm{Corr}(C,E)^{\otimes} := \mathrm{Corr}(C^{\times'}, E^{\times})$

where $C^{\times'} = ((C^{op})^{\sqcup})^{op}$, $C^{\times'} \longrightarrow \mathrm{Fin}_*^{op}$ (NB Not the s.m. category for $(C, \times)$)

- $\mathrm{Ob}(C^{\times'}) = \mathrm{Ob}((C^{op})^{\sqcup}) = \{ (c_i)_{i \in I} \mid I \in \mathrm{Finset},\ c_i \in C \}$
- Morphisms $(I, (x_i)_{i\in I}) \longrightarrow (J, (y_j)_{j \in J})$ is $\begin{cases} \alpha : \langle J \rangle \longrightarrow \langle I \rangle \\ \forall i \in I \quad f_i : x_i \longrightarrow \prod_{j \in \alpha^{-1}(i)} y_j \end{cases}$

and $E^{\times} \subseteq \mathrm{Morph}(C^{\times'})$ lying over the identity and s.t. $\forall i \in I \quad g_i : x_i \longrightarrow y_i \in E$

So explicitely: $\underline{Ob}(\mathrm{Corr}(C,E)^{\otimes}) = \{ (I, (x_i)_{i\in I}) \mid x_i \in ob(C) \}$

Morphism $(I,(x_i)_{i\in I}) \to (J,(y_j)_{j\in J})$ is

$$\begin{tikzcd} & (K,(U_k)_{k\in K}) \arrow[dl, "{(\alpha,(f_k))}"'] \arrow[dr, "\in E^{\times}"] & \\ (I,(x_i)_{i\in I}) & & (J,(y_j)_{j\in J}) \end{tikzcd}$$

So explicately it is
$$\begin{cases} \alpha : \langle I\rangle \longrightarrow \langle K\rangle \\ \forall k\in K \quad f_k : U_k \longrightarrow \prod_{i\in\alpha^{-1}(k)} x_i \\ \mathrm{Id} : \langle K\rangle = \langle J\rangle \\ \forall k\in K \quad g_k : U_k \longrightarrow y_k \in E \end{cases}$$

Meaning $(I,(x_i)_{i\in I}) \to (J,(y_j)_{j\in J})$ is

$$\left\{ \begin{tikzcd} & U_j \arrow[dl, "f_j"'] \arrow[dr, "g_j\in E"] & \\ \prod_{i\in\alpha^{-1}(j)} x_i & & y_j \end{tikzcd} \right\}_{j\in J}$$

and fixed $\alpha : \langle I\rangle \longrightarrow \langle J\rangle$

Rmk Contravariance in Fin.
to cancel
CONTRAVARIANCE LEFT down ↙

PROP [LZ12a] $\mathrm{Corr}(C,E)^{\otimes} \xrightarrow[p]{} \mathrm{Fin}_*$ is a COCARTESIAN FIBRATION, and $\mathrm{Corr}(C,E)^{\otimes}_{\langle I\rangle} \xrightarrow{\sim} \mathrm{Corr}(C,E)^{\times I}$

Rmk The s.m. structure on $\mathrm{Corr}(C,E) \simeq \mathrm{Corr}(C,E)^{\otimes}_{\langle 1\rangle}$ is the coCart lift of $\beta : \langle 2\rangle \longrightarrow \langle 1\rangle$

Consider
$$\begin{tikzcd}
(\langle 2\rangle, (X_1,X_2)) \arrow[r, dashed] \arrow[d, "p"] & ? \\
\langle 2\rangle \arrow[r, "\beta"'] & \langle 1\rangle
\end{tikzcd}$$

Need
$$\begin{tikzcd}
& (\langle 1\rangle, ?) \arrow[dl, "(\beta, ?)"'] \arrow[dr, "(\mathrm{id}, ?)"] & \\
(\langle 2\rangle, (X_1,X_2)) & & (\langle 1\rangle, ?)
\end{tikzcd}$$

and this is
$$\begin{tikzcd}
& (\langle 1\rangle, X_1\times X_2) \arrow[dl, "(\beta, \mathrm{id})"'] \arrow[dr, equal] & \\
(\langle 2\rangle, (X_1,X_2)) & & (\langle 1\rangle, X_1\times X_2)
\end{tikzcd}$$

§4. A 3 FUNCTOR FORMALISM

We said that a 3-functor formalism is LAX S.M. FUNCTOR $\mathrm{Corr}(C,E)^{\otimes} \longrightarrow \mathrm{Cat}_\infty^{\times}$

Thm [2.4.1.7, HA] $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ s.m ∞cat
$\mathcal{D}^{\times} \longrightarrow \mathrm{Fin}_*$ CARTESIAN s.m ∞cat

LAX S.M. FUNCTOR

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr] \arrow[dr] & & \mathcal{D}^{\times} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

$\Longleftrightarrow$

$F : \mathcal{C}^{\otimes} \longrightarrow \mathcal{D}$ functor

s.t.

$$F((I,(x_i)_{i\in I})) \xrightarrow{\sim} \prod_{i\in I} F(\langle i\rangle, x_i)$$

induced by $e_i : \langle I\rangle \longrightarrow \langle i\rangle$

is an equivalence in $\mathcal{D}$

Rmk Easy to see $\Leftarrow$, bc $F(\bigotimes_i X_i) \xleftarrow{F(\bar{\beta}_I)} F((I,(x_i)_{i\in I})) \xrightarrow{\sim} \prod_{i\in I} F(\langle i\rangle, x_i)$

$\bar{\beta}_I$ cocart lift of $\beta_I : \langle I\rangle \longrightarrow \langle 1\rangle$

DEF (C,E) geometric context. Then a <u>3-FUNCTOR FORMALISM</u> is a functor

$$D : \mathrm{Corr}(C,E)^{\otimes} \longrightarrow \mathrm{Cat}_\infty \quad \text{s.t.} \quad D(I,(X_i)_{i\in I}) \xrightarrow{\sim} \prod_{i\in I} D(\langle i\rangle, X_i)$$

is an equivalence.

<u>In particular</u>, objectwise D is determined by the objs

$$\begin{array}{ccc} \mathrm{Corr}(C,E)^{\otimes}_{\langle 1\rangle} & \simeq & \mathrm{Corr}(C,E) \\ \uplus & & \uplus \\ (\langle 1\rangle, Y) & \longleftrightarrow & Y \end{array}$$

$\Rightarrow$ we write $D(X) := D(\langle 1\rangle, X)$

<u>RESULT</u>: D a 3 functor formalism $\rightsquigarrow$ we recover the 3 functors $\otimes, f^*, f^!$ and FUNCTORIALITY + COMPATIBILITY encoded by functoriality of D

(• PROJ FORMULA
• BASE CHANGE)

Indeed • **PULLBACK** $f: X \to Y$ morph in C

$$\Rightarrow \begin{tikzcd} & (\langle 1\rangle, X) \arrow[dl, "(id,f)"'] \arrow[dr, equal] & \\ (\langle 1\rangle, Y) & & (\langle 1\rangle, X) \end{tikzcd} \quad \text{morph } (\langle 1\rangle, Y) \xrightarrow{f} (\langle 1\rangle, X)$$

$$\rightsquigarrow \quad f^* = D(f) : D(Y) \longrightarrow D(X)$$

• **TENSOR PRODUCT** Want $(D(X), \otimes)$ s.m. category.

Let's see how to define $-\otimes- : D(X)\times D(X) \longrightarrow D(X)$

EXTERNAL TENSOR PRODUCT

1) $\boxtimes : D(X)\times D(X) \xleftarrow{\sim} D(\langle 2\rangle, (X,X)) \xrightarrow{D(\bar{\beta})} D(X\times X)$

2) $diag: X \longrightarrow X\times X \Rightarrow diag^* : D(X\times X) \longrightarrow D(X)$

$$\Rightarrow \quad -\otimes- : D(X)\times D(X) \xleftarrow{\sim} D(\langle 2\rangle, (X,X)) \xrightarrow{D(\bar{\beta})} D(X\times X) \xrightarrow{diag^*} D(X)$$

• **EXCEPTIONAL PUSHFORWARD** $f: X \to Y$ in E $\Rightarrow$

$$\begin{tikzcd} & (\langle 1\rangle, X) \arrow[dl, equal] \arrow[dr, "(id,f)"] & \\ (\langle 1\rangle, X) & & (\langle 1\rangle, Y) \end{tikzcd}$$

$$\rightsquigarrow \quad f_! = D(f) : D(X) \longrightarrow D(Y)$$

Compatibility laws $\rightsquigarrow$ functoriality of D

- ACTION of D on a generic morphism:

$$\begin{tikzcd} & (J,(U_j)_j) \arrow[dl, "(\alpha,(f_j)_j)"'] \arrow[dr, "(id,(g_j)_j)"] & \\ (I,(X_i)_{i\in I}) & & (J,(Y_j)_j) \end{tikzcd} \quad \leftrightsquigarrow \quad \left\{ \begin{tikzcd} & U_j \arrow[dl, "f_j"'] \arrow[dr, "g_j"] & \\ \prod\limits_{i\in\alpha^{-1}(j)} X_i & & Y_j \end{tikzcd} \right\}_{j\in J}$$

and

$$\begin{tikzcd} \prod\limits_{i\in I} D(X_i) \simeq \prod\limits_{j\in J}\Big(\prod\limits_{i\in\alpha^{-1}(j)} D(X_i)\Big) \arrow[r, "\prod\limits_{j\in J} \boxtimes f_j^*"'] & \prod\limits_{j\in J} D(U_j) \arrow[r, "\prod\limits_{j\in J}(g_j)_!"'] & \prod\limits_{j\in J} D(Y_j) \\ D(I,(X_i)_{i\in I}) \arrow[u, "\wr"'] & & D(J,(Y_j)_j) \arrow[u, "\wr"'] \end{tikzcd}$$

Some compatibility relation

<u>PROJECTION FORMULA</u> : $f_! A \otimes B \simeq f_!(A \otimes f^* B)$ functorially

$f : X \to Y$ in E i.e. $\underbrace{f_!(-) \otimes -}_{(1)} \simeq \underbrace{f_!(- \otimes f^*(-))}_{(2)}$ as functors $D(X) \times D(Y) \to D(Y)$

$\ldots \mathrm{id}, (f, \mathrm{id}))$
$\ldots \rangle, (Y, Y))$

$$(1) = \circ\, D\left(
\begin{tikzcd}[column sep=small]
 & (\langle 1\rangle, Y \times Y) \arrow[dl, "{(\beta, \mathrm{id})}"'] \arrow[dr, equal] & \\
(\langle 2\rangle, (Y, Y)) & & (\langle 1\rangle, Y \times Y)
\end{tikzcd}
\circ
\begin{tikzcd}[column sep=small]
 & (\langle 1\rangle, Y) \arrow[dl, "\mathrm{diag}"'] \arrow[dr, equal] & \\
(\langle 1\rangle, Y \times Y) & & (\langle 1\rangle, Y)
\end{tikzcd}
\right)$$

$$(2) = D\left(
\begin{tikzcd}[column sep=small]
 & (\langle 2\rangle, (X, X)) \arrow[dl, "{(\mathrm{id}, (\mathrm{id}, f))}"'] \arrow[dr, equal] & \\
(\langle 2\rangle, (X, Y)) & & (\langle 2\rangle, (X, X))
\end{tikzcd}
\circ
\begin{tikzcd}[column sep=small]
 & (\langle 1\rangle, X \times X) \arrow[dl] \arrow[dr, equal] & \\
(\langle 2\rangle, (X, X)) & & (\langle 1\rangle, X \times X)
\end{tikzcd}
\circ
\begin{tikzcd}[column sep=small]
 & (\langle 1\rangle, X) \arrow[dl, "\mathrm{diag}"'] \arrow[dr, equal] & \\
(\langle 1\rangle, X \times X) & & (\langle 1\rangle, X)
\end{tikzcd}
\circ
\begin{tikzcd}[column sep=tiny]
 & (\langle 1\rangle, X) \arrow[dl, equal] \arrow[dr, "f"] & \\
(\langle 1\rangle, X) & & (\langle 1\rangle, Y)
\end{tikzcd}
\right)$$

$\longrightarrow$

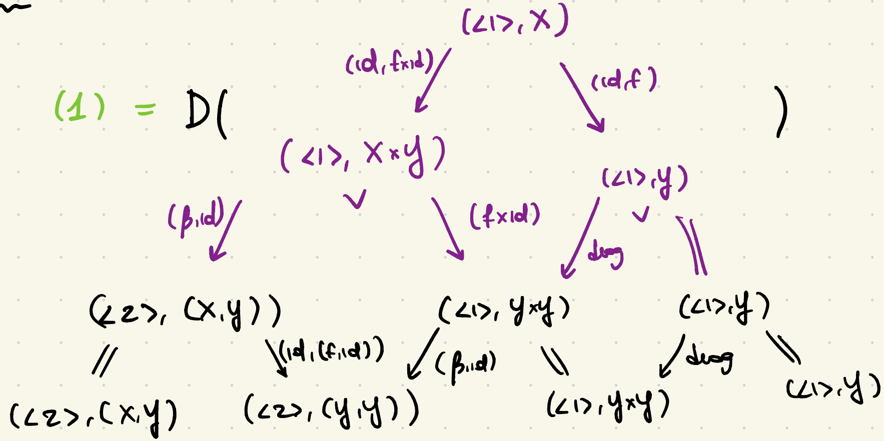
to read in this direction

Now if we look at the compositions of the two morphisms in $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$

They are the SAME !

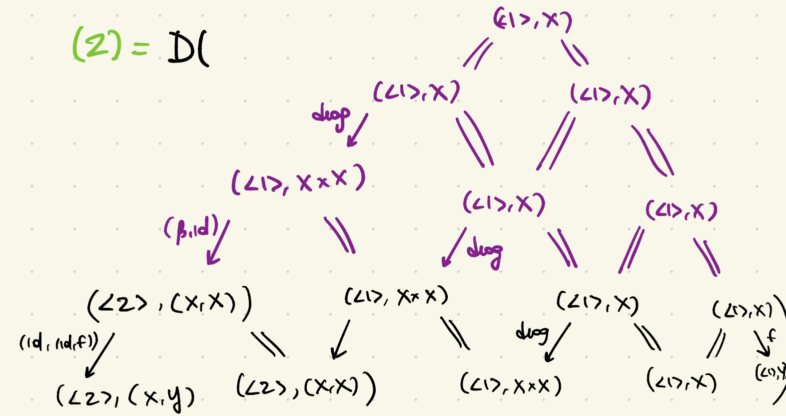
See

$(1) = D($



$)$

$(2) = D($



where first pullback for example is

$$\begin{tikzcd}
X \times X \arrow[d, "f\times id"'] & (y\times y) \underset{y\times y}{\times} (X\times y) = X\times y \arrow[l, dashed] \arrow[d, dashed] \\
y \times y & y \times y \arrow[l]
\end{tikzcd}$$

$$\Rightarrow D\left(\begin{tikzcd}[column sep=small] & (\langle 1\rangle, X) \arrow[dl, "(\beta, id\times f)"'] \arrow[dr, "(id, f)"] & \\ (\langle 2\rangle, X\times y) & & (\langle 1\rangle, y) \end{tikzcd} \right)$$

OK

Similar way we get BASE CHANGE.

<u>Final Remark</u>: $f: X \to Y$ in E $\rightsquigarrow$ $D(Y) \curvearrowright D(X)$ via $D(Y) \times D(X) \xrightarrow{f^* \times id} D(X) \times D(X) \xrightarrow{\otimes} D(X)$

and WANT (in a 3 funct formalism) that $f_!: D(X) \longrightarrow D(Y)$ is $D(Y)$-lin.

<u>This is TRUE</u> bc D is LAX S.M. functor

↳ Indeed: • D lax s.m. → D preserves algs & mods

• $(C^{op})^{\sqcup} \longrightarrow Corr(C,E)^{\otimes}$ s.m. functor

$$\begin{tikzcd} X \arrow[d, "f \text{ in } C"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}$$

X is a module over Y in C^{op}
(as comm algs in $(C^{op}, \sqcup)$)

$$\begin{tikzcd} X \sqcup X \; (= X \times X) \arrow[r, "\Delta"] & X \\ Y \sqcup Y \arrow[u, "f \sqcup f"] \arrow[r, "\Delta"'] \arrow[ur, phantom, "\circlearrowleft"] & Y \arrow[u, "f"'] \end{tikzcd}$$

and $g: X \xrightarrow[\text{in } E]{} X'$ map$/Y$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & Y \end{tikzcd}$$

⇓

induces

$X \longrightarrow X'$ map of Y-mod in $Corr(C,E)$

LAX S.M.
$\Longrightarrow D(g): g_!: D(X) \longrightarrow D(X')$ $\rightsquigarrow$ Now take $X' = Y$.
map of $D(Y)$-mod