(1) Adjunctions between ∞-categories

<u>Def</u> An adjunction is a map $\mathcal{M} \to \Delta^1$ which is both a cartesian an cocartesian fibration

Equivalently, it is an adjunction in the 2-category of ∞-Categories:

$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, a natural transf $FG \to id_{\mathcal{D}}$ and $id_{\mathcal{C}} \to GF$

~~st (FG)F → F and G → (GF)G~~

$$F \to F(GF) = (FG)F \to F \text{ is homotopic to } id_F$$

and

$$G \to (GF)G = G(FG) \to G \text{ is homotopic to } id_G.$$

<u>Proposition</u> left adjoint $\mathcal{C} \to \mathcal{D}$ preserve all colimits that exist in $\mathcal{C}$

Dually right adjoint $\mathcal{D} \to \mathcal{C}$ preserve all limits that exist, in $\mathcal{D}$.

(2) Localization of ∞-categories.

<u>Def</u> A localization of $\mathcal{C}$ is a left adjoint $L : \mathcal{C} \to \mathcal{D}$ whose right adjoint is fully faithful.

<u>Prop</u> $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is fully faithful with essential image the functors $\mathcal{C} \to \mathcal{E}$ sending L-equivalences to equivalences (where L-equivalences are defined as the maps of $\mathcal{C}$ sent to equivalences by L).

(3) Ind-categories κ-regular cardinal

Def A κ-filtered ∞-cat is an ∞-Cat s.t every κ-small diagram $K \longrightarrow \mathcal{C}$ has a cocone.

Prop $\mathcal{C}$ κ-filtered $\Rightarrow$ there is a κ-filtered poset A + a cofinal functor $NA \longrightarrow \mathcal{C}$.

Prop $\mathcal{C}$ κ-filtered iff the colimit functor $\operatorname{colim}_{\mathcal{C}} : \operatorname{Fun}(\mathcal{C}, \mathcal{S}) \longrightarrow \mathcal{S}$ preserves κ-small limits.

Def $\mathcal{C}$ small cat $\operatorname{Ind}_\kappa(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ = ∞-category of functors that are κ-filtered colimits of representables

Prop If $\mathcal{C}$ has κ-small colimits
$\operatorname{Ind}_\kappa(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ is the ∞-category of functors that preserves κ-small limits.

Prop [Universal property] Suppose that $\mathcal{D}$ admits κ-filtered colimits then composition with the Yoneda embedding induces an equivalence

$$\operatorname{Fun}_{\substack{\kappa\text{-colim} \\ \text{preserving}}}(\operatorname{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \longrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

Recognition Theorem let $\operatorname{Ind}_\kappa \mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor such that

(1) $F|_{\mathcal{C}}$ is fully faithful
(2) F preserves filtered colim
(3) The image of F lands in κ-compact objects of $\mathcal{D}$
(4) Any object of $\mathcal{D}$ is a κ-filtered colimit of objects in $\operatorname{Im} F$

Then F is an equivalence.

④ Presentable ∞-cat

Def $\mathcal{C}$ is κ-presentable if it is complete and of the form $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ for $\mathcal{C}_0$ a small ∞-category

~~Prop~~ In fact we may assume that $\mathcal{C}_0$ has κ-small colimits thanks to the following construction

Cns Take $\mathcal{C}^\kappa \subset \mathcal{C}$ to be the κ-compact objects then $\mathcal{C}_0 \subset \mathcal{C}^\kappa$ and $\mathcal{C}^\kappa$ is stable under κ-small colimits we moreover

$$\begin{tikzcd} \mathrm{Ind}_\kappa(\mathcal{C}_0) \arrow[r] \arrow[rr, bend right, "\sim"'] & \mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \arrow[r, "(*)"] & \mathcal{C} \end{tikzcd}$$

I claim that $(*)$ is an equivalence. This follows from the ~~reg~~ recognition theorem of Ind categories.

Moreover $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{P}(\mathcal{C}^\kappa)$ is an accessible localization [Simpson theorem]

⑤ Adjoint theorem

Prop Let $\mathcal{C}$ be presentable, $F: \mathcal{C}^{op} \to \mathcal{S}$ is representable iff it preserves limits

Corollary: ~~$\mathcal{C}$~~ $F: \mathcal{C} \to \mathcal{D}$ preserves colimits $\iff$ F is a left adjoint

Proof $\mathrm{map}(F(-), d)$ is representable for all d $\square$

Proof of the Proposition

Assume F preserves limits then F is determined by its restriction to $\mathcal{C}^\kappa$

$F: (\mathcal{C}^\kappa)^{op} \to \mathcal{S}$ preserves κ-small limits

$\Rightarrow$ this corresponds to an object in $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \xrightarrow{\sim} \mathcal{C}$

Prop $F: \mathcal{C} \longrightarrow \mathcal{S}$ is corepresentable iff it is κ-~~co~~ filtered colimit preserving for some κ (or) accessible + it preserves limits

Coro $G: \mathcal{D} \longrightarrow \mathcal{C}$ is a right adjoint $\Leftrightarrow$ accessible + limit preserving

⑥ Limits and colimits in $\mathcal{P}r^L$

$\mathcal{P}r^L$: presentable ∞-cat + left adjoint functors
$\mathcal{P}r^R$: ———— right adjoint functors

$(\mathcal{P}r^L)^{op} \simeq \mathcal{P}r^R$

Prop presentable categories are stable under the following construction

(1) taking $Fun(K,-)$ with K small

(2) slice and coslice

(3) limits in Cat_∞. Moreover $Forget: \mathcal{P}r^L \longrightarrow Cat_\infty$ preserves limits. Example $\mathcal{D}(X) = \lim_{\operatorname{spec} R \to X} \mathcal{D}(R)$

Theo $\mathcal{P}r^R$ has small limits and $Forget: \mathcal{P}r^R \longrightarrow Cat_\infty$ preserve small limits

Example $Spec = \lim\left(\cdots S_* \xrightarrow{\Omega} S_* \to S_*\right)$ in $\mathcal{P}r^R$ or Cat_∞

$= \operatorname{colim}\left(S_* \xrightarrow{\Sigma} S_* \xrightarrow{\Sigma} S_* \to \cdots\right)$ in $\mathcal{P}r^L$

⑦ Lurie tensor product.

Let $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{P}r^L$

$\mathcal{C}\otimes\mathcal{D} \in \mathcal{P}r^L$ is such that

$$LFun(\mathcal{C}\otimes\mathcal{D}, \mathcal{E}) = Fun'(\mathcal{C}\times\mathcal{D}, \mathcal{E})$$

where Fun' are those functors that preserve colimits in both variables.

<u>Prop</u> $\mathcal{C}\otimes\mathcal{D} \simeq RFun(\mathcal{C}^{op}, \mathcal{D})$

<u>Proof</u> $Fun'(\mathcal{C}\times\mathcal{D}, \mathcal{E})$ ~~$\to LFun(\mathcal{E}, RFun(\mathcal{E},\mathcal{D})^{op})$~~

$\simeq LFun(\mathcal{C}, LFun(\mathcal{D},\mathcal{E}))$

$\simeq LFun(\mathcal{C}, RFun(\mathcal{E},\mathcal{D})^{op})$

$\simeq LFun(\mathcal{C}, LFun(\mathcal{E}^{op},\mathcal{D}^{op}))$

$\simeq LFun(\mathcal{E}^{op}, LFun(\mathcal{C}, \mathcal{D}^{op}))$

$\simeq RFun(\mathcal{E}, RFun(\mathcal{C}^{op},\mathcal{D}))^{op}$ (~~fun~~ image = accessible functors from $\mathcal{E}$ to RFun)

$\to Fun'(\mathcal{C}\times\mathcal{D}, \mathcal{E})) \simeq LFun(RFun(\mathcal{C}^{op},\mathcal{D}), \mathcal{E})$ □

<u>Example</u> K small ∞-cat $\mathcal{P}(K)\otimes\mathcal{D} \simeq RFun(\mathcal{P}(K)^{op},\mathcal{D})$

$\simeq Fun(K^{op},\mathcal{D})$

more generally if K is a small site

$Sh(K)\otimes\mathcal{D} \simeq \mathcal{D}$ valued sheaves on K.

The tensor product makes $\mathcal{P}r^L$ has a sym-monoidal ∞-cat

Moreover $\otimes$ preserves colimits in each variables