ALGEBRAIC MODELS AND MOTIVIC STRUCTURE ON HOMOTOPY TYPES

GEOFFROY HOREL

Abstract. We review some algebraic models for rational or integral homotopy types with the motivation of formalizing what it means for affine algebraic group to act on a homotopy type. We give some applications to the construction of a motivic structure on the homotopy type of algebraic varieties.

This paper is dedicated to the memory of Larry Breen (1944–2023).

1. A formalization of the problem

Let R be a commutative ring. We wish to answer the following vague question.

Question 1.1. How much information about a homotopy type X can be extracted from its cohomology with coefficients in R?

There are many ways to make this question precise. Let us give some examples in the case of ordinary cohomology with $\mathbb{F}_2$-coefficients.

(1) If we think of cohomology simply as a collection of abelian groups, then it is not a fine enough invariant to distinguish $S^1 \vee S^2$ from $\mathbb{RP}^2$.
(2) If we think of cohomology as a graded ring, we can distinguish $S^1 \vee S^2$ from $\mathbb{RP}^2$, but not $S^2 \vee S^3$ from $\Sigma\mathbb{RP}^2$.
(3) If we think of cohomology as a module over the Steenrod algebra, then we can distinguish $S^2 \vee S^3$ from $\Sigma\mathbb{RP}^2$.
(4) Whatever structure we put on the cohomology, it will not be able to distinguish BC_p with p odd from a point.

Let us make some comments. The first three points follow from standard computations in algebraic topology. The last point follows from the fact that the map $BC_p \to pt$ induces an isomorphism on $\mathbb{F}_2$-cohomology. From these facts, we see first that there is no hope of detecting more than the R-localization of the homotopy type from cohomology with R-coefficients. We also see that, to have a good invariant, we should include at least power operations. These power operations arise from the fact that cochains with R coefficients on a space form a commutative algebra in the ∞-category $\mathsf{D}(R)$. At the point set level, this structure is encoded by an action of a model for the E_∞-operad (see [BF04]).

Work by Sullivan, for $R = \mathbb{Q}$, and Mandell, for $R = \mathbb{F}_p$ or $\mathbb{Z}$, has shown that cochains with R coefficients, together with their E_∞-structure, are a fine enough invariant to distinguish nilpotent, finite-type homotopy types up to R-localization (see [Sul77, Man01, Man06]).

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However, the case of $\mathbb{Q}$-coefficients is quite different from the case of $\mathbb{Z}$-coefficients. Over $\mathbb{Q}$, the cochain functor is fully faithful from the category of rational, finite-type spaces to the category of commutative algebras. Over $\mathbb{Z}$, this functor is only faithful. This can be remedied by replacing the target category with something finer, namely cosimplicial binomial rings as in [Hor24], or variants thereof, as in [KSZ23, Ant23].

One motivation for having these algebraic models is that they can be used to formalize what it means for an algebraic group to act on a homotopy type. In particular, in any situation in which the cohomology of a space can be lifted in a sufficiently coherent way to the category of representations of an affine algebraic group, we can obtain an action on the homotopy type, and in particular, on the homotopy groups. This approach is quite classical in rational homotopy theory, and we explain in Section 6 how we can recover the mixed Hodge structures on the homotopy types of algebraic varieties, as well as other motivic structures. Integrally, this is less well-known, and we sketch a construction of a motivic structure on the homotopy type of algebraic varieties in the final section.

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Notations. For A an abelian category, we denote by $\mathsf{D}(\mathsf{A})$ the ∞-categorical derived category. We denote by $\mathsf{D}_{\leqslant 0}(\mathsf{A})$ the full subcategory of coconnective objects (with homological grading convention). We write S for the ∞-category of spaces.

We denote by Map the mapping space in an ∞-category. We usually do not write the name of the ∞-category unless there is a risk of ambiguity.

2. Preliminaries

2.1. **Localization.** Let R be a commutative ring. A space U is R-local if for any $f : X \to Y$ such that $H_*(f, R)$ is an isomorphism, the induced map

$$\mathrm{Map}(Y, U) \to \mathrm{Map}(X, U)$$

is a weak equivalence. The ∞-category of R-local spaces is stable under limits. Bousfield, in [Bou75], has shown that there exists a left adjoint to the inclusion of R-local spaces in the ∞-category of all spaces. We shall denote by $X \mapsto X_R$ this left adjoint. In the particular case of $R = \mathbb{F}_p$, we write X_p instead of $X_{\mathbb{F}_p}$. If X is simply connected and of finite type and R is a subring of $\mathbb{Q}$, then the map

$$X \to X_R$$

induces the canonical map $\pi_i(X) \to \pi_i(X) \otimes R$ on homotopy groups and likewise, the map

$$X \to X_p$$

induces $\pi_i(X) \to \pi_i(X) \otimes \mathbb{Z}_p$. In particular, $\mathbb{Z}$-localization is equivalent to the identity functor when restricted to simply connected homotopy types.

Remark 2.1. In the presence of fundamental group, the situation becomes much more subtle, In particular, there exist acyclic spaces, i.e. non-contractible spaces whose integral homology is the same as the integral homology of the point. The $\mathbb{Z}$-localization of such a space is contractible.

2.2. **Nilpotent spaces.** The following class of spaces will play a key role in what follows.

Definition 2.2. A connected space X is nilpotent of finite type if

(1) Its fundamental group is nilpotent.
(2) The action of the fundamental group on higher homotopy groups is nilpotent.
(3) The integral homology is finitely generated in each degree.

What makes nilpotent finite type spaces useful is that they are built from Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$ with $n \geqslant 1$ using limits of a very explicit form. It follows that many theorems about them can be reduced to the case of Eilenberg-MacLane spaces. This is formalized in Proposition 2.4 below.

Definition 2.3. A convergent tower is a tower of connected spaces

$$\ldots \to X_n \to X_{n-1} \to \ldots \to X_0$$

in which the connectivity of the fiber of $X_i \to X_{i-1}$ is a function of i that tends to infinity.

Proposition 2.4. *Let F and G be two functors from the ∞-category of nilpotent finite type spaces to any ∞-category with limits. Let $\alpha : F \to G$ be a natural transformation. Assume that the following assumptions hold*

(1) The functors F and G preserve finite products.
(2) The functors F and G preserve fiber sequences with simply connected base.
(3) The functors F and G preserve limits of convergent towers.
(4) The natural transformation α is a weak equivalence on Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$ with $n \geqslant 1$.

Then α is a weak equivalence on any finite type nilpotent space.

3. Rational homotopy theory

In rational homotopy theory, we consider the functor of singular cochains

$$X \mapsto C^*(X, \mathbb{Q})$$

from the ∞-category of spaces to the category $\mathsf{CAlg}(\mathsf{D}(\mathbb{Q}))^{\mathrm{op}}$. This functor is a left adjoint and its right adjoint is given by

$$A \mapsto \mathrm{Map}_{\mathsf{CAlg}}(A, \mathbb{Q})$$

Theorem 3.1. [Sul77] *The unit of the adjunction is rationalization for X nilpotent of finite type.*

Proof. Since the functor

$$X \mapsto \mathrm{Map}(C^*(X, \mathbb{Q}), \mathbb{Q})$$

obviously inverts rational equivalences, the unit map can be factored as

$$X \to X_{\mathbb{Q}} \to \mathrm{Map}(C^*(X, \mathbb{Q}), \mathbb{Q})$$

We wish to prove that the map $X_{\mathbb{Q}} \to \mathrm{Map}(C^*(X), \mathbb{Q})$ is a weak equivalence for all nilpotent spaces of finite type. For this, we may apply Proposition 2.4. The first three assumptions are easily verified. Therefore, it suffices to prove the statement for $X = K(\mathbb{Z}, n)$. We consider F_n, the free commutative algebra over $\mathbb{Q}$ generated by $\mathbb{Q}[-n]$. We thus have

$$F_n = \bigoplus_{k \geqslant 0} \mathbb{Q}[-kn]_{\Sigma_k}$$

where the Σ_k action is the kn-th power of the sign representation. From this, it follows that

$$H^*(F_n) \cong \mathbb{Q}[x_n]$$

if n is even and

$$H^*(F_n) \cong \mathbb{Q}[x_n]/x_n^2$$

if n is odd (with $|x_n| = -n$). On the other hand, applying the Serre spectral sequence to the fiber sequence

$$K(\mathbb{Z}, n) \to * \to K(\mathbb{Z}, n+1)$$

and reasoning by induction on n, it is straightforward to prove that $H^*(K(\mathbb{Z}, n), \mathbb{Q})$ is abstractly isomorphic to $H^*(F_n)$ as a graded commutative algebra.

Now, observe that we have an isomorphism

$$H^n(K(\mathbb{Z}, n), \mathbb{Q}) \cong [K(\mathbb{Z}, n), K(\mathbb{Q}, n)] \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathbb{Q})$$

we can thus pick a class in $H^n(K(\mathbb{Z}, n), \mathbb{Q})$ corresponding to the canonical inclusion $\mathbb{Z} \to \mathbb{Q}$. By universal property of the free commutative algebra construction, this induces a map of commutative algebras

$$F_n \to C^*(K(\mathbb{Z}, n), \mathbb{Q}).$$

inducing an isomorphism in cohomological degree n. This together with the computation of the cohomology of both sides implies that this map is a quasi-isomorphism. It follows that there is a weak equivalence

$$\mathrm{Map}(C^*(K(\mathbb{Z}, n), \mathbb{Q}), \mathbb{Q}) \simeq \mathrm{Map}(F_n, \mathbb{Q}) \simeq K(\mathbb{Q}, n)$$

(the second equivalence comes form free forgetful adjunction between commutative algebras and cochain complexes). Putting everything together, we deduce that the composite

$$K(\mathbb{Z}, n) \to \mathrm{Map}(C^*(K(\mathbb{Z}, n)), \mathbb{Q}) \simeq K(\mathbb{Q}, n)$$

is the canonical map and therefore, the induced map on rationalization is a weak equivalence as desired. □

Remark 3.2. This theorem is only "one half" of Sullivan's work in homotopy theory. The other half is given by the theory of minimal models that makes rational homotopy theory into a very computable theory. We will not say anything about that in these notes although the theory of minimal models will play a role in the proof of Proposition 4.2. Let us also mention that in [Sul77], Sullivan introduces another functor from spaces to commutative algebras in $\mathsf{D}(\mathbb{Q})$ called Ω^*_{PL}. It has the property that, at the point set level, it is given by a strictly commutative differential graded algebra (as opposed to $C^*(-, \mathbb{Q})$ which is merely an E_∞-algebra). Nevertheless, the two functors are equivalent and Theorem 3.1 holds for both functors.

4. The rational affinization of homotopy types

For R a commutative ring, we denote by Aff_R the category of affine schemes over R. To avoid set theoretic problems, we implicitly restrict to the category of affine schemes whose ring of functions has bounded cardinality so that the resulting category is essentially small.

Given a homotopy type X, we obtain a functor

$$A \mapsto \mathrm{Map}(C^*(X, \mathbb{Q}), A)$$

from the category of ordinary $\mathbb{Q}$-algebras to the ∞-category of spaces. Equivalently, this is a presheaf on the category of affine schemes over $\mathbb{Q}$. We denote it by $X^{aff}_{\mathbb{Q}}$.

Let R be a commutative ring. The additive group is simply the functor $\mathsf{Aff}_R^{\mathrm{op}} \to \mathsf{S}$ sending $\mathrm{Spec}(B)$ to the abelian group B. This is an abelian group object in $\mathrm{Fun}(\mathsf{Aff}_{\mathbb{Q}}^{\mathrm{op}}, \mathsf{S})$ and therefore, it can be delooped arbitrarily. We denote by $K(\mathbb{G}_a, i)$ its i-fold delooping. Explicitly, it is given by

$$\mathrm{Spec}(B) \mapsto K(B, i)$$

Definition 4.1. [Toë06] Let R be a commutative ring. The ∞-category of *affine stacks* over R is the smallest full subcategory of $\mathrm{Fun}(\mathsf{Aff}_R^{\mathrm{op}}, \mathsf{S})$ that is stable under limits and containing the stacks $K(\mathbb{G}_a, i)$ for all $i \geqslant 0$.

Proposition 4.2. *If X is connnected and finite type, then $X_{\mathbb{Q}}^{aff}$ is an affine stack over $\mathbb{Q}$.*

Proof. This can be viewed from Sullivan's theory of minimal models. For C a cochain complex, we denote by $\mathrm{Sym}(C)$ the free commutative algebra on C. Observe that the stack $K(\mathbb{G}_a, n)$ is represented by $\mathrm{Sym}(\mathbb{Q}[-n])$.

The theory of minimal models asserts that the commutative algebra $C^*(X, \mathbb{Q})$ can be built as transfinite composition of pushouts of maps of the form $\mathbb{Q} \to \mathrm{Sym}(\mathbb{Q}[-n])$ and $\mathrm{Sym}(\mathbb{Q}[-n]) \to \mathbb{Q}$. It follows that X^{aff} is the limit of a transfinite tower in which each map is a pullback of $pt \to K(\mathbb{G}_a, n)$ or $K(\mathbb{G}_a, n) \to pt$. □

5. Affine group actions on rational homotopy types

Let $\mathbf{k}$ be a field of characteristic zero. We denote by Spec the functor

$$\mathrm{Spec} : \mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbf{k}))^{\mathrm{op}} \to \mathrm{Fun}(\mathsf{Aff}_{\mathbf{k}}^{\mathrm{op}}, \mathsf{S})$$

sending A to $S \mapsto \mathrm{Map}_{\mathsf{CAlg}(\mathsf{D}(\mathbf{k}))}(A, S)$. Observe that, if A is an ordinary $\mathbf{k}$-algebra, then the functor $\mathrm{Spec}(A)$ takes values in discrete (i.e. 0-truncated) homotopy type, and coincides with the ordinary spectrum functor in algebraic geometry.

Observe also that the functor Spec sends coproducts in the ∞-category $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbf{k}))$ to products. Moreover, coproducts in $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbf{k}))$ are simply given by tensor products. It follows that if H is a cogroup object in $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbf{k}))$ then $\mathrm{Spec}(H)$ is a group object. Likewise if A has a coaction of H, then $\mathrm{Spec}(A)$ has an action of $\mathrm{Spec}(H)$.

Let G be an ordinary affine algebraic group over $\mathbf{k}$. That is $G = \mathrm{Spec}(H)$ with H a commutative Hopf algebra over $\mathbf{k}$. By taking the functor of points, G can be viewed as a group object in $\mathrm{Fun}(\mathsf{Aff}_{\mathbf{k}}^{\mathrm{op}}, \mathsf{Set})$ and therefore also in the ∞-category $\mathrm{Fun}(\mathsf{Aff}_{\mathbf{k}}^{\mathrm{op}}, \mathsf{S})$.

Now, denote by $\mathsf{Rep}(G)$ the category of representations of G. This is simply the category of comodules over the Hopf algebra H. Explicitly, a comodule is a $\mathbf{k}$-vector space V equipped with a coaction map

$$V \to V \otimes_{\mathbf{k}} H$$

satisfying the expected coassociativity and counitality axioms. The fact that H is a Hopf algebra and not merely a coalgebra implies that the category $\mathsf{Rep}(G)$ admits a symmetric monoidal structure. The tensor product of two objects M and N is simply the tensor product $M \otimes_{\mathbf{k}} N$ equipped with the following structure map

$$M \otimes N \to (M \otimes H) \otimes (N \otimes H) \cong (M \otimes N) \otimes (H \otimes H) \to (M \otimes N) \otimes H$$

where the last map is induced by the multiplication map of H.

If $A \in \mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbf{k}))$ admits a lift $\tilde{A}$ in $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathsf{Rep}(G)))$, then $\tilde{A}$ may be viewed as a cocaction of H on A. It follows that $\mathrm{Spec}(A)$ has an action of the group G.

Let us specialize even further to $\mathbf{k} = \mathbb{Q}$. Let X be an object of S. Assume that there is an action of G on $X_{\mathbb{Q}}^{aff}$ in the ∞-category $\mathrm{Fun}(\mathsf{Aff}_{\mathbb{Q}}^{\mathrm{op}}, \mathsf{S})$. According to the previous discussion, this can be equivalently seen as a lift of $C^*(X,\mathbb{Q})$ to an object of $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathsf{Rep}(G)))$. Assume further, the existence of a G fixed point on $X_{\mathbb{Q}}^{aff}$, i.e. a G-equivariant map

$$x : \mathrm{Spec}(\mathbb{Q}) \to X^{aff}$$

where $\mathrm{Spec}(\mathbb{Q})$ is equipped with the trivial G-action. In this case, the rational homotopy groups of X based at x obtain canonically the structure of G-representations. Explicitly, given a $\mathbb{Q}$-algebra R, a pointed homotopy class of maps $\alpha : S^n \to X_{\mathbb{Q}}^{aff}(R)$ and an element $g \in G(R)$, we define $g.\alpha$ as the following composite:

$$S^n = S^n \times * \xrightarrow{\mathrm{id}\times 1_{G(R)}} S^n \times G(R) \xrightarrow{\alpha\times\mathrm{id}} X_{\mathbb{Q}}^{aff}(R) \times G(R) \to X_{\mathbb{Q}}^{aff}(R)$$

6. Applications

6.1. **Mixed Hodge structure.** Let X be a complex algebraic variety and X^{an} be its underlying complex analytic space. Then the commutative algebra $C^*(X^{an},\mathbb{Q})$ can be promoted to a commutative algebra in mixed Hodge structures as explained in [CH20, Section 6]. This was also in Drew's thesis ([Dre15], unpublished). The abelian category of mixed Hodge structure is Tannakian and thus can be identified with the category of finite dimensional representations of an affine group scheme $G = \mathrm{Spec}(H)$ (see [DM82]). It follows from the discussion of the previous section that the affine stack $(X^{an})^{aff}$ is naturally equipped with an action of the group G. In particular, if $x \in X^{an}$ is a point, then the homotopy groups of X^{an} based at that point can be promoted to mixed Hodge structures. This structure has been first constructed by Morgan in [Mor78] in the case of smooth varieties (see also [Nav87, Hai87, Cir15] for generalizations of this construction).

6.2. **Motivic structure.** Likewise, let $\mathbf{k} \subset \mathbb{C}$ be a field. We can endow homotopy type of algebraic varieties with a motivic structure. For this, it suffices to find a factorization of the Betti realization functor as

$$\mathsf{Var}_{\mathbf{k}}^{\mathrm{op}} \to \mathsf{CAlg}(\mathsf{D}(\mathsf{Rep}(G_{mot}))) \to \mathsf{CAlg}(\mathsf{D}(\mathbb{Q}))$$

wher G_{mot} is some motivic Galois group.

There are several ways of making this precise. Historically, the first construction of this type where restricted to mixed Tate motives. In this case, there is an affine group scheme G_{MTM} and a factorization as above but we need to restrict to varieties whose motives is mixed Tate (i.e. in the smallest triangulated category of the category of motives containing the Tate twists). This kind of construction is described in [Del89] or [DG05].

Alternatively, one can use the motivic Galois group of Nori $G_{mot}^N = \mathrm{Spec}(H^N)$ (whose construction is recalled in [HMS17]). The required factorization can be found in [CGAdS17]. We also refer to [Gar06] that proves a similar result using a different approach. Finally, if one is willing to use a motivic Galois group which is derived (i.e. represented by a derived affine group scheme), then there is a canonical factorization as above through representations of Ayoub's motivic Galois group $G_{mot}^A = \mathrm{Spec}(H^A)$ (defined in [Ayo14]) with H^A a dg-Hopf algebra. Such a construction can be found in [Iwa20]. Note also that the two constructions are related as $H^N = H^0(H^A)$ by the main result of [CGAdS17].

7. Mandell and Toën's theorem

If we move from rational to integral homotopy theory, the functor $X \mapsto C^*(X,\mathbb{Z})$ with values in $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbb{Z}))$ is no longer fully faithful. Nevertheless, thanks to a famous theorem of Mandell, this functor knows everything about nilpotent finite type homotopy types. In order to state this theorem we introduce the finite loop space $\Lambda_f X$ of a space X which is defined so that the following square is cartesian

$$\begin{array}{ccc} \Lambda_f X & \longrightarrow & \prod_p \Lambda X_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X_p)_{\mathbb{Q}} \end{array}$$

In this diagram Λ denotes the free loop space functor: $\Lambda U := \mathrm{Map}(S^1, U)$. The right vertical map is the composite

$$\prod_p \Lambda X_p \to \prod_p X_p \to (\prod_p X_p)_{\mathbb{Q}}$$

where the first map is evaluation of a loop at a chosen base point and the second map is rationalization.

Observe that, for X nilpotent and of finite type, this diagram can be embedded in Sullivan's arithmetic square as follows

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p \\ \downarrow & & \downarrow{\scriptstyle cst} \\ \Lambda_f X & \longrightarrow & \prod_p \Lambda X_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X_p)_{\mathbb{Q}} \end{array}$$

where *cst* denotes the inclusion of the constant loops in the free loop space. In particular, there is a canonical map $X \to \Lambda_f X$.

Theorem 7.1. [Man06] *The functor $X \mapsto C^*(X, \mathbb{Z})$ from S^{op} to $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbb{Z}))$ is a left adjoint, the unit of this adjunction is the canonical map*

$$X \to \Lambda_f X$$

when X is nilpotent and of finite type.

Proof. The proof starts by studying the unit map

$$X \to \mathrm{Map}_{\mathsf{CAlg}(\mathsf{D}(\mathbb{F}_p))}(C^*(X, \mathbb{F}_p), \overline{\mathbb{F}}_p).$$

Mandell shows that this map is a p-completion for nilpotent finite type spaces. This is an application of Proposition 2.4 and an explicit description of $C^*(K(\mathbb{Z}, n), \mathbb{F}_p)$. It follows that the unit map

$$X \to \mathrm{Map}_{\mathsf{CAlg}(\mathsf{D}(\mathbb{F}_p))}(C^*(X, \mathbb{F}_p), \mathbb{F}_p)$$

can be identified with $X \to X_p^{Frob}$ where X_p^{Frob} is the fixed point space for the action of the Frobenius. It is not hard to check that this action must be trivial so that $X_p^{Frob} \simeq \Lambda X_p$. Then Mandell shows that the canonical map

$$\mathrm{Map}_{\mathsf{CAlg}(\mathsf{D}(\mathbb{Z}_p))}(C^*(X, \mathbb{Z}_p), \mathbb{Z}_p) \to \mathrm{Map}_{\mathsf{CAlg}(\mathsf{D}(\mathbb{F}_p))}(C^*(X, \mathbb{F}_p), \mathbb{F}_p)$$

is an equivalence for X nilpotent and of finite type. This can be viewed again as an application of Proposition 2.4 which reduces it to the case of $X = K(\mathbb{Z}, n)$.

Finally, one uses the fact that the square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & (\prod_p \mathbb{Z}_p) \otimes \mathbb{Q} \end{array}$$

is cartesian in the category $\mathsf{CAlg}(\mathsf{D}(\mathbb{Z}))$. Mapping $C^*(X, \mathbb{Z})$ into this square we get the desired result. □

Remark 7.2. This theorem implies that the functor $C^*(-, \mathbb{Z})$ is quite far from being fully-faithful. For example $[S^2, S^2] \cong \mathbb{Z}$ while $[C^*(S^2, \mathbb{Z}), C^*(S^2, \mathbb{Z})] \cong \mathbb{Z} \oplus \prod_p \mathbb{Z}_p$.

One important corollary of Mandell's theorem is the following.

Theorem 7.3. *Let X and Y be nilpotent of finite type. Then X is weakly equivalent to Y if and only if $C^*(X, \mathbb{Z})$ is weakly equivalent to $C^*(X, \mathbb{Z})$ in $\mathsf{CAlg}(\mathsf{D}_{\leqslant 0}(\mathbb{Z}))$.*

Proof. A nilpotent finite type space fits in a cartesian square of the following form

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X_p)_{\mathbb{Q}}. \end{array}$$

From this it follows easily that there exists a map $\Lambda_f X \to X$ such that the composed map

$$X \to \Lambda_f X \to X$$

is homotopic to the identity. □

Toën has shown that a similar theorem holds in a more rigid category. For X a simplicial set, we write $\mathbb{Z}^X$ the cosimplicial commutative ring given by $\mathbb{Z}^{X_k}$ in cosimplicial degree k. If we invert the weak equivalences of simplicial sets and the quasi-isomorphisms of cosimplicial rings, this induces a functor from the ∞-category S^{op} to the ∞-category cCRing obtained from the 1-category of cosimplicial commutative rings by inverting quasi-isomorphisms.

Theorem 7.4. [Toë20] *The functor $X \mapsto \mathbb{Z}^X$ is a left adjoint. The unit of this adjunction is the map*

$$X \to \Lambda_f X$$

for X nilpotent and of finite type. Moreover two nilpotent finite type spaces X and Y are weakly equivalent if and only of $\mathbb{Z}^X$ and $\mathbb{Z}^Y$ are weakly equivalent as cosimplicial commutative rings.

From an ∞-categorical perspective, the category cCRing may seem a bit ad hoc. In fact Toën observe that the ∞-category cCRing is equivalent to the ∞-category of affine stacks (Definition 4.1) over $\mathbb{Z}$ through the spectrum functor:

$$A^\bullet \mapsto (B \mapsto \mathrm{Map}_{\mathsf{cCRing}}(A^\bullet, B))$$

where B is viewed as a constant cosimplicial commutative ring. In particular, Toën defines the affinization of a homotopy type X denoted X^{aff} to be the affine stack over $\mathbb{Z}$ corresponding to $\mathbb{Z}^X$ through the above equivalence. We sum-up Toën's work in the following Theorem.

Theorem 7.5. [Toë20] *Let X be a nilpotent space of finite type. Then X^{aff} is an affine stack with the following properties.*

- *Its space of $\mathbb{Q}$-points is $X_{\mathbb{Q}}$.*
- *Its space of $\overline{\mathbb{F}}_p$-points is X_p.*
- *Its space of $\mathbb{F}_p$-points is ΛX_p.*
- *Its space of $\mathbb{Z}$-points is $\Lambda_f X$.*

8. Binomial rings and homotopy theory

Recall that, if R is a commutative ring, r is an element of R and $n \in \mathbb{N}$, the generalized binomial coefficient $\binom{r}{n}$ is defined by

$$\binom{r}{n} := \frac{r(r-1)\dots(r-n+1)}{n!}$$

It is an element of $R \otimes \mathbb{Q}$.

Definition 8.1. A binomial ring is a commutative ring R whose underlying abelian group is torsion free and which is such that, for all $r \in R$ and $n \in \mathbb{N}$, the generalized binomial coefficient $\binom{r}{n}$ belongs to R.

Remark 8.2. It is often useful to know the following alternative characterization. A binomial ring is a torsion-free commutative ring R in which, for every element $r \in R$ and every prime p, the number p divides $r^p - r$. See [Ell06, Theorem 4.1].

Example 8.3. Any subring of the rationals (in particular $\mathbb{Z}$) is binomial. Any $\mathbb{Q}$-algebra is a binomial. The ring $\mathbb{Z}_p$ of p-adic integers is binomial for any p.

As a last example, consider the ring $\mathrm{Num}[x_1, \dots, x_n] \subset \mathbb{Q}[x_1, \dots, x_n]$ of numerical polynomials in n variables. Recall that a numerical polynomial is a polynomial with rational coefficients which takes integer values when evaluated on integers. It can be shown that this ring is binomial. This is in fact the free binomial ring on n generators.

The category of binomial rings is a full subcategory of the category of commutative rings. It is in fact monadic and comonadic over the category of commutative rings. In particular, binomial rings are stable under limits and colimits inside commutative rings. It follows that, if R is a binomial and X is a simplicial set, the cosimplicial commutative ring R^X is a cosimplicial binomial ring. Let us denote by cBRing the ∞-category obtained form the 1-category of cosimplicial binomial rings by inverting quasi-isomorphisms.

Theorem 8.4. [Hor24] *The functor $X \mapsto R^X$ induces a left adjoint functor from the ∞-category S to the ∞-category $\mathsf{cBRing}^{\mathrm{op}}$. The unit of this adjunction is homotopic to the identity map $X \to X$ for X nilpotent of finite type. If X is finite type, the unit is given by the map $X \to \mathbb{Z}_\infty X$ to the Bousfield-Kan $\mathbb{Z}$-completion.*

Proof. The proof again is based on Proposition 2.4. We are essentially reduced to proving that $\mathbb{Z}^{K(\mathbb{Z},n)}$ is quasi-isomorphic as a cosimplicial binomial ring to $F_n := \mathrm{Bin}(\mathbb{Z}[-n])$, the free cosimplicial binomial ring on a class of degree $-n$. This is done by Toën in [Toë20]. In [Hor24], we give a different proof based on a reduction to the case $n = 1$ which is then a standard homological algebra calculation. □

Let $\mathbb{G}_a^{bin}$ denote the functor from BRing to Ab sending B to the underlying abelian group of B. Similarly to Toën, we can make the following definition.

Definition 8.5. The category of affine binomials stacks is the smallest full subcategory of $\mathrm{Fun}(\mathsf{BRing}, \mathsf{S})$ that is stable under small limits and contains $K(\mathbb{G}_a^{bin}, i)$ for all $i \geqslant 0$.

Theorem 8.6. *The assignment $X \mapsto X^{bin}$ is a fully faithful functor from the ∞-category of nilpotent finite type spaces to the ∞-categeory of binomial stacks. Moreover, for X nilpotent and of finite type. We have*

- *The space of $\mathbb{Z}$-points of X^{bin} is X.*
- *For R a subring of $\mathbb{Q}$, the space of R-points of X^{bin} is X_R.*
- *The space of $\mathbb{Z}_p$-points of X is X_p.*

Remark 8.7. We conclude this section with some review of the literature on algebraic models for integral homotopy types. First of all, there exists three versions of Theorem 8.4. One due to the author which is the one stated in this document, one due to Antieau (see [Ant23]) and one due to Kubrak-Shuklin-Zakharov (see [KSZ23]). In spirit these three theorems are saying the same thing but their target category are different. In [KSZ23], the target category is the ∞-category of coconnective derived binomial rings, in [Ant23], the target is coconnective derived lambda-rings with the data of a trivialization of the Frobenius lift. It is very believable that these three ∞-categories are equivalent although no proof seems to exist at the moment.

In a different direction, if one is willing to work over the sphere spectrum rather than the integers, there is a model due to Yuan that takes values in commutative ring spectra with trivialization of the Frobenii (see [Yua23]). There is also a model due to to Heuts in terms of Tate coalgebras (see [Heu21]). The definition of Tate coalgebras is inductive and not so easy to grasp but the advantage of working coalgebraically is that one can get rid of the finite type assumption. In this direction, we should mention the recent preprint [BB24] of Bachman and Burklund that proves a p-adic version of Mandell's theorem using coalgebras. This allows them to remove the finite type hypothesis.

9. The problem of schematization

This idea originates in Grothendieck's "Pursuing stacks" [Gro83]. Grothendieck conjectures that, to a homotopy type X, there should be an associated schematization X^{sch} which would be a (higher) stack with the following list of properties.

1. Its space of $\mathbb{Z}$-points is the underlying homotopy type of X.
2. The homotopy groups of the R points should be naturally an R-module.
3. Its space of R-points is the R-completion/localization for reasonable rings R.
4. The cohomology of X should coincide with the of X^{sch} with coeffients in $\mathbb{G}_a$.
5. If $X = K(\mathbb{Z}, n)$, then X^{sch} should be $K(\mathbb{G}_a, n)$.

Of course the knowledgeable reader will observe that these requirements are contradictory. Indeed the p-completion of $K(\mathbb{Z}, n)$ is $K(\mathbb{Z}_p, n)$ and not $K(\mathbb{F}_p, n)$. Nevertheless Grothendieck's idea was to take item (5) for a fact and construct the required object in general using "Postnikov dévissage". This dream was put to an end after a phone call to Illusie in which Grothendieck learned about the work of Breen on extensions of the additive group (see [Bre78]). The problem one has to deal with is that there are no non-trivial central extensions of $\mathbb{Z}$ by itself (in other words $H^2(\mathbb{Z}, \mathbb{Z}) = 0$) whereas there are non-trivial central extensions of $\mathbb{G}_a$ by itself. One of them is given by the truncated Witt vectors W_2.

Grothendieck then goes on to propose a way to fix of his idea. Instead of paraphrasing, let us just cite him:

> *This example brings near one plausible "reason" why the expected comparison statement about discrete and schematic linearization could not reasonably hold true, and in particular why we shouldn't expect discrete and schematic Hochschild cohomology (for group schemes over $\mathbb{Z}$ such as $\mathbb{G}_a$ or successive extensions of such) to give the same result. Namely, the latter is computed in terms of cochains which are polynomial functions with coefficients in $\mathbb{Z}$, whereas there exist polynomial functions with coefficients in $\mathbb{Q}$ (not in $\mathbb{Z}$) which, however, give rise to integer-valued functions on the group of integer-valued points (...) Thus, the hope still remains that a sweeping comparison theorem for discrete versus "schematic" linearization might hold true, provided it is expressed in such a way that the "schematic models" we are working with should be built up with "schemes" (of sorts) described in terms of spectra not of polynomial algebras $\mathbb{Z}[t]$ and tensor powers of these, but rather of "binomial algebras" $\mathbb{Z}\langle t\rangle$ built up with the binomial expressions above, and tensor powers of such.*

As we have explained in section 7, Toën in [Toë06, Toë20], constructs what he calls the affinization of homotopy type. This construction can be viewed as the best possible approximation to what Grothendieck has in mind that remains within the world of classical algebraic geometry. Our construction X^{bin} is meant to be a precise version of what Grothendieck suggests in the citation above.

10. Motivic structure on the integral homotopy type of algebraic varieties

Let $\mathbf{k} \subset \mathbb{C}$. Recall from [CGAdS17], that there exists an affine algebraic group over the integer $G^N_{mot}(\mathbf{k}) = \mathrm{Spec}(H^N(\mathbf{k}))$ with $H^N(\mathbf{k})$ a commutative Hopf algebra whose representations are exactly Nori motives. The key observation is the following.

Proposition 10.1. *The underlying commutative ring of $H^N(\mathbf{k})$ is a binomial ring.*

Proof. First of all $H^N(\mathbf{k})$ is flat so it is in particular torsion free. Moreover, it can be shown that $H^N(\mathbf{k}) \otimes \mathbb{F}_p$ is the ring of continuous functions on the absolute Galois group of k:

$$H^N(\mathbf{k}) \otimes \mathbb{F}_p \cong C^0(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}), \mathbb{F}_p)$$

which immediately implies what we need, thanks to Remark 8.2. □

Conjecture 10.2. *Let $\mathcal{X}$ be a smooth algebraic variety over $\mathbf{k}$. Let $X \in \mathsf{S}$ be the Betti realization of $\mathcal{X}$. Then X^{bin} is naturally equipped with an action of $G_{mot}(\mathbf{k})$ with the following properties*

(1) After base change to $\mathbb{Q}$ this action specializes to the classical motivic structure on the rational homotopy type from subsection 6.2.

(2) After base change to $\mathbb{Z}_p$, this action recovers the Galois action on the p-complete homotopy type

Let us sketch how this conjecture can be proved. We have a cartesian square in cBRing given by

$$\begin{array}{ccc} \mathbb{Z}^X & \longrightarrow & \hat{\mathbb{Z}}^X \\ \downarrow & & \downarrow \\ \mathbb{Q}^X & \longrightarrow & (\mathbb{Q} \otimes \hat{\mathbb{Z}})^X \end{array}$$

where $\mathbb{Z} = \prod_p \mathbb{Z}_p$ is the profinite completion of the ring of integers. We shall construct the required action on $\mathbb{Z}^X$ by gluing together an action on the other three corners of the square.

First of all the cosimplicial binomial ring $\mathbb{Z}_p^X$ can be given an action of $\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$. Indeed, by Artin's comparision theorem, $\mathbb{Z}_p^X$ is quasi-isomorphic to pro-étale sheaf cohomology of $\mathcal{X} \times_{\mathbf{k}} \overline{\mathbf{k}}$ with coefficients in $\mathbb{Z}_p$. In particular, this can be constructed as a limit of a diagram of cosimplicial binomial rings. Taking the product over all primes, this should give the desired action on the top right corner. On the other hand, we have seen in subsection 6.2 how to construct an action of $G^N_{mot}(\mathbf{k}) \times_{\mathbb{Z}} \mathbb{Q}$ on $X^{bin} \times_{\mathbb{Z}} \mathbb{Q}$. Finally one should check that the two structures are compatible on the bottom right corner through the canonical map of Hopf algebras

$$H^N(\mathbf{k}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to C^0(\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}), \mathbb{Q}_p)$$

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