HOMOTOPY TRANSFER AND FORMALITY

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Abstract. In [CH17, CH18], the second author and Joana Cirici proved a theorem that says that given appropriate hypotheses, n-formality of a differential graded algebraic structure is equivalent to the existence of a chain-level lift of a homology-level degree twisting automorphism using a unit of multiplicative order at least n.

Here we give another proof of this result of independent interest and under slightly different hypotheses. We use the homotopy transfer theorem and an explicit inductive procedure in order to kill the higher operations. As an application of our result, we prove formality with coefficients in the p-adic integers of certain dg-algebras coming from hyperplane and toric arrangements and configuration spaces.

Introduction

An algebraic structure A (e.g. an associative algebra, a commutative algebra, an operad, etc.) in the category of chain complexes is said to be formal if it is connected to its homology $H_*(A)$ by a zig-zag of quasi-isomorphisms that preserve the algebraic structure. There are many interesting examples from a variety of arenas, including rational homotopy theory [Sul77, FHT01], Kontsevich formality [Kon03, Tam03], Kähler manifolds [DGMS75], string topology of complex projective spaces with integral coefficients [BB17, Theorem 4.3], etc.

Let us generically use the term algebra to refer to any type of algebraic structure such as those mentioned above; we assume that the structure operations are all degree zero. For any algebra A, and any unit α of the base ring, one can construct an automorphism σ_α of $H_*(A)$ that is given in homological degree n by multiplication by α^n. By a standard homotopical algebra reasoning, if A happens to be formal, this automorphism can be lifted to an endomorphism of A (or at least an endomorphism of a cofibrant replacement of A).

It was observed in the introduction of [DGMS75] (see also the last section of [Pet14]) that the converse should be true if α is of infinite order. The intuition is the following : if such an endomorphism exists at the level of chains, then any higher Massey product has to be compatible with this action but then we see that they have to be zero because they intertwine multiplication by α^n with multiplication by α^m with $n \neq m$. As stated, this is not rigorous because the vanishing of all higher Massey products is not a sufficient condition to insure formality. In fact Deligne–Griffiths–Morgan–Sullivan proved their result using a different method. Sullivan proved a statement of this kind for differential graded algebras in characteristic zero [Sul77, Theorem 12.7] which was generalized to other algebraic structures by Guillén Santos–Navarro–Pascual–Roig [GSNPR05, Theorem 5.2.4]. The method in

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both cases uses a chain level filtration and does not explicitly pass through Massey products.

One of our goals in this paper is to give a method for proving formality by making more precise the intuition explained in the previous paragraph. We use the fact that the Massey products are the shadow of the P_∞-structure on the homology of the algebra. Unlike the Massey products, the P_∞-structure is not uniquely defined but it is unique in a suitable homotopical manner. Moreover, it contains all the homotopical information of the algebra. Our main result is the following :

Main Theorem. *Let P be an operad in the category of R-modules. Let (A, d, m) be a P-algebra in the category of differential graded R-modules such that the chain complex $(H_*(A, d), 0)$ can be written as a homotopy retract of (A, d). Let α be a unit in R and let $\hat{\sigma}$ be an endomorphism of (A, d, m) such that the induced map on $H_*(A, d)$ is the degree twisting by α (see Definition 0.1).*

- *If $\alpha^k - 1$ is a unit of R for $k \leq n$, then (A, d, m) is n-formal as a P-algebra.*
- *If $\alpha^k - 1$ is a unit of R for all k, then (A, d, m) is formal as a P-algebra.*

The assumption that $(H_*(A, d), 0)$ can be written as a homotopy retract of (A, d) allows us to apply the homotopy transfer theorem (Theorem 1.14). It is automatic is R is a field or more generally if A and $H_*(A)$ are degreewise projective and R is a hereditary ring (a ring is hereditary if a submodule of a projective module is projective). Let us mention that Dedekind rings (in particular principal ideal domains) are hereditary. These conditions can be weakened a little further. See Remark 2.11.

The main improvement over the kind of classical results of [Sul77] and the improvement of [GSNPR05] is the extension to give n-formality results outside characteristic zero fields. The method of proof in both references relies on a filtration that fails as soon as α does not have infinite order.

One reason this result is interesting is because the theory of étale cohomology gives examples of algebra with automorphisms of this type. Let us explain this with a simple example. We take A to be the algebra $C^*(\mathbf{P}^n_{\mathbb{C}}, \mathbb{Q}_\ell)$ of singular cochains on the complex projective space with $\mathbb{Q}_\ell$-coefficients. Standard results of étale cohomology imply that this algebra is quasi-isomorphic to the algebra $B = C^*_{et}(\mathbf{P}^n_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ of étale cochains of the projective space over the algebraic closure of $\mathbb{Q}$. Since the projective space is actually defined over $\mathbb{Q}$, the algebra B has an action of the absolute Galois group of $\mathbb{Q}$. We can pick a lift σ of the Frobenius of $\overline{\mathbb{F}}_p$ in the absolute Galois group of $\mathbb{Q}$ and it can be shown that σ acts on $H^{2k}(\mathbf{P}^n_{\mathbb{C}}, \mathbb{Q}_\ell)$ by multiplication by p^k. We are thus exactly in the situation of the theorem above and we deduce that A is formal.

We could apply the same strategy with the algebra of cochains of complex projective space with $\mathbb{Z}_\ell$-coefficients. In that case, the Frobenius lift still acts in degree $2k$ by multiplication by p^k. However, contrary to what happens in characteristic zero, $p^{\ell-1} - 1$ is not a unit in $\mathbb{Z}_\ell$. The above theorem lets us conclude that A is $\ell - 2$ formal (it is in fact $2\ell - 4$ formal by the variant 3.1).

A similar suite of results was proven in the papers [CH17] and [CH18] by Cirici and the second author. There the method used was different and relied on deep results in abstract homotopy theory, most notably, Hinich's recent result comparing Lurie's ∞-categorical approach to algebras over an operad with the model categorical approach (see [Hin15]). Here the methods used are comparatively easier and more explicit. We can in fact write an inductive formula for a formality quasi-isomorphism. Moreover, we are able to improve one of the results of [CH18] by removing a simple connectivity hypothesis and allowing the coefficients ring to be more general than a field. From this we obtain a result of partial formality for complement of hyperplane arrangements and toric arrangements with coefficients in the p-adic integers that we believe is new. Let us mention however, that not all of the results of [CH18] can be recovered from the methods of our paper. Most notably, the result of $(p-2)$-formality of the little disks operad with coefficients in $\mathbb{F}_p$ proved in [CH18, Theorem 6.7] is not a consequence of our main theorem.

Structure of the paper. In section 1 we review mostly standard conventions, definitions, and facts about operadic homotopy algebra. We briefly review operads and cooperads, algebras and coalgebras, coderivations, homotopy algebras, and homological perturbation. The only things that are non-standard are the following:

(1) we use the symbol $\triangleleft$ instead of $\circ$ for the composition product of $\mathbb{S}$-modules and $\mathbb{N}$-modules, and
(2) we use the terminology *component* of a homotopy algebra in a non-standard way—see Terminology 1.9 and Remark 1.10.
(3) we define *n-formality* in Definition 1.16 and connect it to formality in Propositions 1.17 and 1.18.

Then Section 2 constitutes the proof of the main theorem. The method is to construct a sequence of isomorphisms of homotopy algebras which witness the coherent vanishing of successively more and more of the higher structure operations.

Section 3 presents a simple algebraic variant of the main theorem. Both this variant and the main theorem are applied in Section 4 to yield the following examples.

(1) Formality of complex algebraic varieties whose mixed Hodge structure is pure of some weight,
(2) Formality of the little disks operad,
(3) Formality of complements of hyperplane arrangements,
(4) Formality of complements of toric arrangements and
(5) Coformality of the space of configurations of points in Euclidean space.

To the best of our knowledge, the results obtained with integral coefficients in examples (3), (4) and (5) are new.

Conventions. Fix a commutative ground ring R. All tensor products are taken over R unless otherwise specified. When working with symmetric operads we insist that all prime numbers are invertible in R (i.e. R is a $\mathbb{Q}$-algebra). The symmetric group on the set $\{1, \dots, n\}$ is denoted $\mathbb{S}_n$.

Definition 0.1. Let V be a graded R-module. Let α be a unit in R. The *degree twisting* by α, denoted σ_α, is the linear automorphism of V which acts on the degree n homogeneous component of V via multiplication by α^n.

1. Reminder and conventions on operadic homotopy algebra

1.1. **Operads.** A $\mathbb{N}$-*module* is a collection $\{P(n)\}$ indexed by $n \geq 0$ of chain complexes over R. A $\mathbb{S}$-*module* P is a collection $\{P(n)\}$ indexed by $n \geq 0$ of right (R-linear, differential graded) $\mathbb{S}_n$-representations. The index n is called the *arity*. When we are working with $\mathbb{S}$-modules, we insist that our ground ring is a field of characteristic zero (so that we can, e.g., identify invariants and coinvariants of symmetric group actions).

Maps of $\mathbb{N}$-modules (respectively $\mathbb{S}$-modules) are collections of (equivariant) chain maps. There are monoidal products on $\mathbb{N}$-modules and $\mathbb{S}$-modules[1] defined as follows:

$$(P \triangleleft Q)(n) = \bigoplus_{k=0}^{\infty} \bigoplus_{n_1+\cdots+n_k=n} P(k) \otimes Q(n_1) \otimes \cdots \otimes Q(n_k).$$

$$(P \triangleleft Q)(n) = \bigoplus_{k=0}^{\infty} P(k) \otimes_{R[\mathbb{S}_k]} \bigoplus_{n_1+\cdots+n_k=n} Q(n_1) \otimes \cdots \otimes Q(n_k) \otimes_{R[\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_k}]} R[\mathbb{S}_n].$$

The unit I has a rank one free R-module in $I(1)$ and the zero representation elsewhere. We suppress the associator isomorphisms throughout.

An *operad* is a monoid in the monoidal category of $\mathbb{S}$-modules. A *cooperad* is a comonoid in this monoidal category. A *non-symmetric operad* (respectively *cooperad*) is a monoid (*comonoid*) in the monoidal category of $\mathbb{N}$-modules. A *coaugmentation* of a (possibly non-symmetric) cooperad C is a cooperad map from the trivial cooperad to C; an augmentation of a (possibly non-symmetric) operad P is an operad map from P to the trivial operad. We will always work with the subcategories of *reduced augmented* operads and *reduced coaugmented* cooperads, and will henceforth suppress these adjectives in our terminology. Here "reduced" means that the underlying $\mathbb{N}$-module or $\mathbb{S}$-module has 0 in arity 0.

Remark 1.2. Essentially everything we do works equally well in the symmetric and non-symmetric case, so we use the same symbol for both products; the reader should interpret it as appropriate for the context.

The "reduced" assumption (starting our indexing at 1 and not 0) can be weakened. This is a common technical restriction to make certain sums finite. Other restrictions (typically an auxiliary filtration with appropriate properties) ensuring this kind of finiteness suffice just as well with mild changes. However, the examples that arise most commonly do not need this extra generality.

It may also be possible to weaken the characteristic zero assumption for $\mathbb{S}$-modules. This would be much more interesting in the sense that algebras over symmetric operads arise often in characteristic p. However, the requisite changes appear to be much more substantial and it is not entirely clear that everything would work.

1.3. **Algebras and coalgebras.** There is a fully faithful functor from the category of chain complexes over R to the category of either $\mathbb{N}$-modules or $\mathbb{S}$-modules which takes a chain complex V to the object with V in arity 0 and the zero complex in all other arities. In a slight abuse, we will write V for the image of V under this functor as well, hoping it causes little confusion. For such an object V we write $Q(V)$ as shorthand for the $\mathbb{N}$-module or $\mathbb{S}$-module $Q \triangleleft^{ns} V$ or $Q \triangleleft V$, whichever is appropriate. This is also necessarily concentrated in arity zero.

An *algebra* over the operad P is a left P-module on an $\mathbb{S}$-module in the image of this fully faithful inclusion of chain complexes. That is, it is a chain complex V

[1]These products are often denoted $\circ$ in the literature. We avoid this because of the potential for confusion.

equipped with a map $P(V) \to V$ which satisfies the usual associativity and unitality constraints. Similarly, a *coalgebra* over the cooperad C is a left C-comodule on an $\mathbb{S}$-module in the image of this fully faithful inclusion of chain complexes. That is, it is a chain complex V equipped with a map $V \to C(V)$ which satisfies the usual coassociativity and counitality constraints.

Fix a cooperad C with structure map Δ and counit ϵ. The cofree conilpotent C-coalgebra on V is the coalgebra $C(V)$ with structure map induced by Δ. By abuse, we typically also use the notation Δ for this structure map

$$C(V) \xrightarrow{\Delta} (C \triangleleft C)(V) \cong C(C(V)).$$

There will be several further times when we abuse notation like this, using the symbol of a map f for the map obtained by taking the monoidal product of f with some identity map or other.

By the universal property of being cofree, given a chain map $f : C(V) \to W$, there is a unique extension to a map F of C-coalgebras $C(V) \to C(W)$. Explicitly, F is given by the composite

$$C(V) \xrightarrow{\Delta} C(C(V)) \xrightarrow{f} C(W)$$

where Δ is the comonoidal structure map of C.

1.4. **Coderivations.** Given a chain map $m : C(V) \to V$, there is another useful extension of m, this time not as a coalgebra map but as a coderivation. To discuss this, we first recall the linearization of the monoidal product $\triangleleft$(see, e.g., [LV12, 6.1] for more details). We describe the non-symmetric case for ease of notation.

The linearization of $\triangleleft$ is easiest to describe in terms of a trifunctor on $\mathbb{N}$-modules. Given three $\mathbb{N}$-modules P, Q, and R, the product $P \triangleleft (Q; R)$ consists of the R-linear summands of $P \triangleleft (Q \oplus R)$, i.e.,

$$(P \triangleleft (Q; R))(n) = \bigoplus_{k=1}^{\infty} \bigoplus_{i=1}^{k} \bigoplus_{n_1+\cdots+n_k=n} P(k) \otimes Q(n_1) \otimes \cdots \otimes R(n_i) \otimes \cdots \otimes Q(n_k).$$

Again we abbreviate $P \triangleleft (Q; R)$ as $P(Q; R)$ if both Q and R are in the image of the inclusion from chain complexes to $\mathbb{N}$-modules. It doesn't make sense to ask about "associativity" but this trifunctor satisfies the following compatibility relations:

$$(P \triangleleft Q) \triangleleft (R; S) \cong P \triangleleft (Q \triangleleft R; Q \triangleleft (R; S)) \tag{1}$$

$$(P \triangleleft (Q; R)) \triangleleft S \cong P \triangleleft (Q \triangleleft S; R \triangleleft S). \tag{2}$$

Now we can define the linearization of $\triangleleft$ as $P \triangleleft_{(1)} R = P \triangleleft (I; R)$. Explicitly we have

$$\begin{aligned}(P \triangleleft_{(1)} R)(n) &= (P \triangleleft (I; R))(n) \\ &= \bigoplus_{k=1}^{\infty} \bigoplus_{i=1}^{k} P(k) \otimes I(1) \otimes \cdots \otimes R(n-k+1) \otimes \cdots \otimes I(1) \\ &\cong \bigoplus_{k=1}^{n+1} \bigoplus_{i=1}^{k} P(k) \otimes R(n-k+1).\end{aligned}$$

The linearization is not associative but rather preLie in general. We will not need the preLie compatibility but rather two other relations which follow from Equations (1) and (2):

$$(P \triangleleft Q) \triangleleft_{(1)} R \cong P \triangleleft (Q; Q \triangleleft_{(1)} R) \tag{3}$$

$$(P \triangleleft_{(1)} Q) \triangleleft R \cong P \triangleleft (R; Q \triangleleft R). \tag{4}$$

These isomorphisms and the associator isomorphism for $\triangleleft$ together satisfy the appropriate analogues of the pentagon relation, and so we may safely suppress them, assuming a unique natural isomorphism between any two parenthesizations that are equivalent by a chain of (modified) such associators.

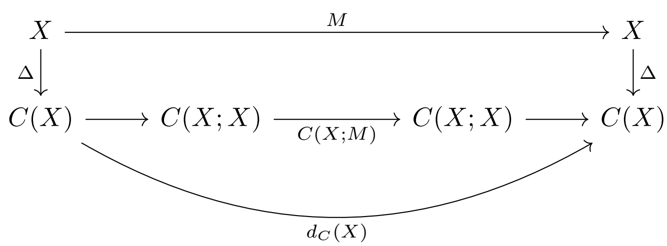
The linearization $P \triangleleft_{(1)} R$ is a direct summand of $P \triangleleft R$ with complement given in terms of similar formulas with either zero or more than one entry from R. A map $O \to Q \oplus R$ induces a map $P \triangleleft O \to P \triangleleft (Q; R)$, and likewise a map $Q \oplus R \to O$ induces a map $P \triangleleft (Q; R) \to P \triangleleft O$.

Now returning to our cooperad C, there is a linearized coproduct from the map $C \xrightarrow{\epsilon, \mathrm{id}_C} I \oplus C$ as follows:

$$C \xrightarrow{\Delta} C \triangleleft C \to C \triangleleft_{(1)} C$$

which we denote $\Delta_{(1)}$.

Now given a C-coalgebra X, a linear map $M : X \to X$ is a *coderivation* of X if the composition along the top and right side of the following diagram is equal to the sum of the composition along the left side and the two choices on the bottom:



where the unmarked arrows are induced by the diagonal $X \to X \oplus X$ and the fold map $X \oplus X \to X$.

The coderivations of the free C-coalgebra $C(V)$ are in bijection with linear maps $C(V) \to V$. Given a coderivation M, one gets a map $m : C(V) \to V$ by projecting to the cogenerators:

$$C(V) \xrightarrow{M} C(V) \to V.$$

In the other direction, given a map $m : C(V) \to V$, one obtains a linear map $C(V) \to C(V)$ via adding $d_C(V)$ to the composition

$$C(V) \xrightarrow{\Delta_{(1)}} (C \triangleleft_{(1)} C)(V) \cong C(V; C(V)) \xrightarrow{m} C(V; V) \to C(V).$$

Here the isomorphism is via Equation (3) and the unmarked arrow is induced by the fold map of V. It is a tedious diagram chase to verify that this indeed gives a coderivation.

There is a similar linearization $\triangleleft_{(1)}$ in the symmetric case which we will not write down explicitly and all of the statements in this section work symmetrically with the minimal requisite changes.

We will also use the following elementary observation about a situation relating a map and its linearization.

Lemma 1.5. *Let C be a cooperad and V a chain complex, and suppose given a map $f : C(V) \to V$. Further suppose that $f_0 = \mathrm{id}_V$ and $f_i = 0$ for $1 \le i < n$. Then the projection*

$$C_N(V) \xrightarrow{\Delta} C(C(V)) \xrightarrow{f} C(V) \to C_{N-n}(V)$$

and the projection

$$C_N(V) \xrightarrow{\Delta_{(1)}} C(V; C(V)) \xrightarrow{f_n} C(V; V) \to C_{N-n}(V)$$

coincide for $n > 0$.

Proof. Applying Δ lands in a sum over partitions of $n = i_1 + \cdots + i_k$ into summands represented by tensor products of the form

$$C_{N-n}(k) \otimes C_{i_1} \otimes \cdots \otimes C_{i_k}$$

and f vanishes on all summands except those with a single index of value n and all other indices of value 0. The sum of such terms is then the projection of $C \triangleleft C$ to $C \triangleleft (C_0; C_n)$. ☐

1.6. **Homotopy algebras.** The *coaugmentation coideal* $\overline{C}$ is the linear cokernel of the coaugmentation $I \to C$. There is a somewhat technical definition of (local) *conilpotence* for a cooperad with respect to a coaugmentation. See, e.g., [LV12, 5.8.5] or [LGL18, 4.1].

The *cobar functor* Ω from conilpotent cooperads to operads takes the cooperad C to the free operad on $\bar{C}[1]$, equipped with a differential that combines the internal differential of C and the comonoid structure of C. See, e.g., [LV12, 6.5.2]. This works equally well for nonsymmetric cooperads and operads. A *twisting morphism* from a conilpotent cooperad C to an operad P is an operad morphism from $\Omega(C)$ to P; the twisting morphism is *Koszul* if it induces an isomorphism on homology groups.

Assumption 1.7. Let P be an operad over R, concentrated in degree zero. Let C be a conilpotent cooperad over R, with coaugmentation coideal $\bar{C}$ concentrated in strictly positive degree. Let $\kappa : C \to P$ be a Koszul twisting morphism.

Given Assumption 1.7, one model for the category of *strongly homotopy P-algebras*, or *P_∞-algebras* is the category of cofree conilpotent C-coalgebras. In this model, we encode a P_∞-algebra structure on (V, d) via a degree -1 linear map $C(V) \to V$ which takes the identity to d and satisfies relations which imply that the corresponding coderivation of the cofree conilpotent coalgebra $C(V)$ squares to zero. We pass back and forth between these representations by using a capital letter for the coderivation and the corresponding lowercase letter for the projection to the cogenerators. E.g., (V, M) and (V, m) are both notation for the same P_∞ algebra structure on V and $M : C(V) \to C(V)$ is the C-coalgebra coderivation extending $m : C(V) \to V$.

Similarly, we encode a P_∞-morphism from V to W (leaving the structures implicit) via a linear map $C(V) \to W$ satisfying relations which imply that the corresponding coalgebra morphism intertwines the coderivations. Again, we use capital letters for the coalgebra maps and lower case letters for the projections to the cogenerators.

Remark 1.8. Given P concentrated in degree zero, the image BP of P under the bar functor satisfies the conditions of Assumption 1.7: it is conilpotent, the coaugmentation coideal is concentrated in strictly positive degrees, and it comes with a canonical Koszul twisting morphism to P. So if we don't care about the details of the particular choice of model for the category of strongly homotopy P-algebras, we need only begin with an operad over R concentrated in degree zero. Since many practitioners have preferred models, especially in the classically Koszul case, we have explicitly recorded the required conditions on the governing cooperad C and its relation to P.

Terminology 1.9. Given a map (say, m or f) from $C(V)$ to V, we use a subscript to indicate the further decomposition with respect to the homological degree of C, and call the resulting maps *components*.

So for instance $m_i : C_i(V) \to V$ is the *ith component* of the P_∞ algebra m and $f_i : C_i(V) \to W$ is the *ith component* of a P_∞-morphism f.

Remark 1.10 (Warning). This does not in general coincide with the normal use of *component* in operadic algebra. Often the ith component would be the component in *arity* i in C, but for us it is the component in *homological degree* i in C.

Note also that this does not coincide with the homological degree of the operations. Since M is supposed to be of degree -1, the operations of form m_i have degree $i-1$. For maps F (of degree zero) the f_i operations indeed have degree i.

Example. The classical examples (the "three graces") and other classical Koszul operads are all examples. In particular, the following examples work.

(1) Let P be the associative operad and C the shifted coassociative cooperad. Then P_∞-algebras are A_∞ algebras with the standard definitions and the example fits into this framework.
(2) Let P be the Lie operad and C the shifted cocommutative cooperad. Then P_∞-algebras are L_∞ algebras with the standard definitions and the example fits into this framework.
(3) Let P be the commutative operad and C the shifted coLie cooperad. Then P_∞-algebras are C_∞ algebras with the standard definitions and the example fits into this framework.
(4) All of these are subsumed by the following. Let P be a (classically) Koszul operad concentrated in degree zero. Then P_∞-algebras with the standard definitions fit into this framework.

Example. Everything we will do works with only the evident requisite changes for colored operads. So another example that works is for P the colored operad whose algebras are non-unital Markl operads. The operad P is concentrated in degree zero so it fits into this framework. The operad P is also classically Koszul so there is an explicit small model for homotopy operads as P_∞-algebras [vdL03].

1.11. **Homological perturbation and formality.** Given a differential graded P-algebra (A, d, m), there is an induced P-algebra structure on the homology $H(A, d)$, because the operations making up m are all chain maps. However, in general the induced structure $(H(A, d), 0, m_*)$ is not equivalent to the original P-algebra (A, d, m).

Definition 1.12. We call the differential graded P-algebra (A, d, m) *formal* if it is equivalent to the induced structure $(H(A, d), 0, m_*)$, i.e., if there exists a differential graded P-algebra $(\hat{A}, \hat{d}, \hat{m})$ and a zig-zag of maps of differential graded P-algebras inducing isomorphisms on homology:

$$(A, d, m) \xleftarrow{\sim} (\hat{A}, \hat{d}, \hat{m}) \xrightarrow{\sim} (H_*(A, d), 0, m_*).$$

Another way to say this is that (A, d, m) and $(H(A, d), 0, m_*)$ represent the same object in the homotopy category of differential graded P-algebras. This notion goes back to [DGMS75]. The category of P_∞-algebras is one model for this homotopy category, and that gives another way to view formality. In the dg category this was understood in special cases; possibly the earliest explicit general statement is Markl's.

Theorem 1.13 ([Mar04, (M1),(M3)]). *Let R be a characteristic zero field. A differential graded P-algebra is formal if and only if there is a map of P_∞-algebras between (A, d, m) and $(H_*(A, d), 0, m_*)$ inducing an isomorphism on homology.*

Even in the absence of formality and working over a general ground ring, as long as (A, d) and $(H_*(A, d), 0)$ are homotopy equivalent chain complexes, there is always a way to compress the data of (A, d, m) to the homology, via the so-called homological perturbation lemma or transfer theorem. Perturbation methods are a classical tool in algebraic topology. They were first used for A_∞-algebras by Kadeishvili [Kad80]. For algebras over more general operads, see [Mar04, LV12, Ber14].

Theorem 1.14 (Transfer theorem). *Let (A, d, m) be a differential graded P-algebra such that the chain complex $(H_*(A, d), 0)$ can be written as a homotopy retract of (A, d). Then there exist*

(1) *a transferred P_∞-algebra structure m^t with zero differential on $H_*(A, d)$ extending the induced P-algebra structure on the homology and*
(2) *quasi-inverse P_∞ quasi-isomorphisms between the P_∞-algebras (A, d, m) and $(H_*(A, d), 0, m^t)$ extending a given homotopy retraction between (A, d) and $(H_*(A, d), 0)$.*

Note that such a homotopy retraction always exists over a field but this also holds if both A and $H_*(A, d)$ are degreewise projective and the base ring R is hereditary. Indeed if this is the case, we have an epimorphism $Z_n(A) \to H_n(A)$ from the group of n-cycles to the n-th homology group. This epimorphism splits since $H_n(A)$ is projective. We can thus write $Z_n(A)$ as the direct sum $H_n(A) \oplus B_n(A)$. Moreover, we can identify $A_n/Z_n(A)$ with $B_{n-1}(A)$ which is projective. Therefore the epimorphism $A_n \to B_{n-1}(A)$ induced by the differential also splits and we have a splitting $A_n \cong B_n(A) \oplus H_n(A) \oplus B_{n-1}(A)$. The construction of the homotopy retraction then works exactly as in the case of fields.

Until the end of this section, we let (A, d, m) be a differential graded P-algebra such that the chain complex $(H_*(A, d), 0)$ can be written as a homotopy retract of (A, d) so that the transfer theorem applies.

Corollary 1.15. *The algebra (A, d, m) is formal if and only if $(H_*(A, d), 0, m^t)$ and $(H_*(A, d), 0, m_*)$ are isomorphic as P_∞-algebras.*

Definition 1.16. Let n be a positive integer, we say that a differential graded P-algebra (A, d, m) is n-formal if $(H_*(A, d), 0, m^t)$ is isomorphic to a P_∞-algebra $(H_*(A, d), 0, m)$ with $m_i = 0$ for i in the range $1 \leq i \leq n$.

We have the following two propositions that show that n-formality can sometimes imply formality.

Proposition 1.17. *Let (A, d, m) be a differential graded P-algebra. Assume that $H_i(A, d) = 0$ for i outside of the interval $[0, n]$. Then, (A, d, m) is n-formal if and only if it is formal.*

Proof. Indeed, the components m_i with $i > n$ of any P_∞-structure on $H_*(A, d)$ have to be zero for degree reasons. □

There is also a somewhat more involved version for cohomologically graded algebraic structures. In order to keep consistent conventions throughout, we phrase it in terms of homologically graded structures concentrated in nonpositive degrees; statements with nonnegative cohomological grading can be obtained by negating indices.

Proposition 1.18. *Let (A, d, m) be a differential graded P-algebra. Let j be an integer, n and q a positive integers. Suppose that*

(1) *for all i, the component C_i is concentrated in arity at least $i + j$, and*
(2) *$H_i(A, d) = 0$ for i outside $[n - q(n + j - 1), -q]$.*

Then (A, d, m) is n-formal if and only if it is formal.

The most common applications occur in the case where

- The operad P is trivial in arity 0 and 1, in which case C can be chosen so that $j = 2$, and
- The integer q is specified to be 2 (the *simply connected* case)

in which case the restriction is that $H_i(A, d) = 0$ for i outside $[-n - 2, -2]$.

Proof. Applying m_i to elements in degree at most $-q$ yields something in degree at most $i - (i + j)q$. If $i \geq n + 1$ then

$$i - (i + j)q = i(1 - q) - jq \leq n - nq - jq - q + 1$$

so the output is outside of the support of $H_*(A, d)$. □

2. Extending formality inductively

Throughout this section we will assume given the data (P, C, κ) of Assumption 1.7.

The following lemma is the technical core of the argument that we will use.

Lemma 2.1. *Let V be a graded R-module and $\sigma_\alpha : V \to V$ be the degree twisting morphism by α where α is a unit in R which is such that $\alpha^n - 1$ is also a unit. Suppose that V (viewed as a chain complex with trivial differential) is equipped with*

(1) *a P_∞-algebra structure $C(V) \xrightarrow{m} V$ which vanishes on $C_i(V)$ for $i = 0$ and i in the range $2 \leq i < n + 1$ and*
(2) *a P_∞-automorphism s of (V, m) which is equal to the automorphism σ_α on $V \cong C_0(V) \to V$ and which vanishes on $C_i(V)$ in the range $1 \leq i < n$.*

Then there exist:

(1) *a P_∞-algebra structure m' on V which vanishes on $C_i(V)$ for $i = 0$ and i in the range $2 \leq i < n + 2$,*

(2) *a P_∞-automorphism s' of (V, m') which is equal to the automorphism σ_α of V on $V \cong C_0(V) \to V$ and which vanishes on $C_i(V)$ in the range $1 \leq i < n+1$, and*
(3) *a P_∞-isomorphism f from (V, m) to (V, m') which is equal to id_V on $V \cong C_0(V) \to V$, vanishes on $C_i(V)$ for $i \notin \{0, n\}$, and intertwines the P_∞-automorphisms s and s'.*

In words, given m which is formal up to degree n components and s which is formal up to degree $n-1$ components, we can build isomorphic data m' and s' with the formality range improved by 1.

We will prove this lemma constructively in stages, assuming throughout that n is at least 1. For the remainder of the section, assume as given the data in the hypotheses of Lemma 2.1.

Definition 2.2. We begin by defining a map $f_n : C_n(V) \to V$ as follows. On the homogeneous degree $n+N$ component of $C_n(V)$ (i.e., the component with homological degree n in C and degree N in tensor powers of V) we act by $\frac{1}{\alpha^{N+n} - \alpha^N}$ times s_n (the degree n component of the automorphism s, which has the desired domain and codomain). The hypotheses on α ensure that the coefficient is well-defined.

Now define a linear map $f : C(V) \to V$ as

$$f_i : C_i(V) \to V = \begin{cases} \mathrm{id}_V & i = 0 \\ f_n & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Then f defines a map of coalgebras $F : C(V) \to C(V)$. With the appropriate understanding of the bracket, we could write $f_n \coloneqq [s_0^{-1}, s_n]$.

Lemma 2.3. *the map F is invertible, and writing f^{-1} for the projections of F^{-1} to the cogenerators V we have the following properties on its components:*

$$f_i^{-1} = \begin{cases} \mathrm{id} & i = 0 \\ 0 & 1 \leq i < n \\ -f_n & i = n \end{cases}$$

(we make no claim for $i \geq n$).

Proof. Invertibility follows because the degree zero component is invertible. To get the formulas for F^{-1}, we note that the projection of the composition of F^{-1} and F is given via

$$C(V) \xrightarrow{\Delta} C(C(V)) \xrightarrow{f)} C(V) \xrightarrow{f^{-1}} V$$

which must be id_V on the $C_0(V)$ summand and 0 otherwise. For $i = 0$ this forces the component f_0^{-1} to be $(f_0)^{-1} = \mathrm{id}_V$. For $1 \leq i < n$, since the component f_j vanishes in the given range, the only term of the composition that a priori survives is the component of form $f_i^{-1}(C(f_0))$—so this term vanishes as well. Since f_0 is the identity, this means that f_i^{-1} vanishes. For $i = n$, the two terms that survive are the components f_n and f_n^{-1}, whose sum is supposed to vanish. □

Now we can define the structures M' and S' via conjugation.

Definition 2.4. Given the construction of Definition 2.2 and Lemma 2.3, define a map $M' : C(V) \to C(V)$ of degree -1 and a map $S' : C(V) \to C(V)$ of degree 0 as follows.

$$M' = FMF^{-1} \qquad\qquad S' = FSF^{-1}.$$

Lemma 2.5. *The data of Definition 2.4 satisfies the following properties:*

- *M' is a P_∞-algebra structure on V,*
- *S' is a P_∞-automorphism of (V, M'), and*
- *F is a P_∞-isomorphism between (V, M) and (V, M').*

Proof. Since M squares to zero, the map M' is a degree -1 square zero coderivation of the cofree C-coalgebra, and thus a P_∞-algebra structure on V. Likewise, from the properties of S, the map S' is a differential graded C-coalgebra map with respect to M' and thus a P_∞ automorphism of the P_∞ algebra (V, M'). By construction, the coalgebra map F intertwines the coderivations M and M' and the automorphisms S and S'. □

It remains to be seen that m' and s' have the promised description. For the next two proofs we will employ a simplified notation

Notation 2.6. Given maps f and g from $C(V)$ to V, we will write $f \triangleleft g$ for the composition

$$C(V) \xrightarrow{\Delta} C(C(V)) \xrightarrow{g} C(V) \xrightarrow{f} V$$

and $f \triangleleft_{(1)} g$ for the composition

$$C(V) \xrightarrow{\Delta_{(1)}} C \triangleleft_{(1)} C(V) \xrightarrow{g} C(V) \xrightarrow{f} V.$$

We call this *schematic notation*.

Lemma 2.7. *The P_∞-algebra structure M' has component m'_1 equal to m_1. The component m'_i vanishes on $C_i(V)$ for $i = 0$ and for i in the range $2 \leq i < n + 2$.*

Proof. First note that the composition $m' : C(V) \to V$ has two contributions. There is a part from m of the form

$$C(V) \to (C \triangleleft_{(1)} C) \triangleleft C(V) \xrightarrow{f^{-1}} (C \triangleleft_{(1)} C)(V) \xrightarrow{m} C(V) \xrightarrow{f} V \tag{5}$$

where the first map is induced by the comonoid map and linearized comonoid map of C. The other part comes from the internal differential of C and is of the form

$$C(V) \to C \triangleleft C(V) \xrightarrow{f^{-1}} C(V) \xrightarrow{d_C} C(V) \xrightarrow{f} V. \tag{6}$$

We have the following facts about these compositions:

- the decomposition map of C respects degree,
- the 0th components of f and f^{-1} are the identity,
- the ith components of f and f^{-1} vanish for $1 \leq i < n$,
- the operation m_0 vanishes, and
- the differential d_C reduces degree by 1 and vanishes on C_1.

These facts immediately imply that the ith component of the composition (5) coincides with m_i for $0 \leq i < n + 1$ and that the ith component of the composition (6) vanishes for i in the same range. It remains to show that $m'_{n+1} = 0$, for which we will need a more complicated argument.

Let us review the condition that s is a an automorphism of (V,m). This is the condition that $MS=SM$, which can be written as the equality of the composition

$$C(V)\to C(C(V))\xrightarrow{s} C(V)\xrightarrow{m} V; \tag{7}$$

and the sum of the compositions

$$C(V)\to C(V;C(V))\xrightarrow{m} C(V;V)\to C(V)\xrightarrow{s} V \tag{8}$$

and

$$C(V)\xrightarrow{d_C} C(V)\xrightarrow{s} V. \tag{9}$$

We focus first on the $n+1$st component of the composition (7). We know m_i vanishes for $i=0$ and $2\le i<n+1$. Therefore the only terms that survive use m_1 or m_{n+1}. By Lemma 1.5, the m_1 term is

$$C(V)\xrightarrow{\sigma_\alpha} C(V)\to C_1(V;C_n(V))\xrightarrow{s_n\sigma_\alpha^{-1}} C_1(V;V)\to C_1(V)\xrightarrow{m_1} V.$$

The other term in (7) uses first $s_0=\sigma_\alpha$ on each argument and then applies m_{n+1}.

There are only two non-vanishing terms in the compositon (8). One involves applying first m_{n+1} and then s_0. Since C_0 is trivial, this term can be rewritten

$$C_{n+1}(V)\xrightarrow{m_{n+1}} V\xrightarrow{s_0} V.$$

Moreover, because m_{n+1} is of homological degree n, this is α^n times

$$C_{n+1}(V)\xrightarrow{s_0=\sigma_\alpha} C_{n+1}(V)\xrightarrow{m_{n+1}} V.$$

The other term involves acting first by m_1 and then s_n. We can write s_n as the composition $(s_n\sigma_\alpha^{-1})\sigma_\alpha$ and then σ_α commutes with m_1 which is homological degree 0.

Finally, the composition (9) has only a single term, first acting by d_C and then by s_n. Again $(s_n=s_n\sigma_\alpha^{-1})\sigma_\alpha$, and since d_C acts only on the C factor, this is the composition where first we apply σ_α, then d_C, and finally $s_n\sigma_\alpha^{-1}$.

So schematically we can write the equation as follows:

$$\begin{aligned}(m_1\triangleleft_{(1)} s_n\sigma_\alpha^{-1})\sigma_\alpha+m_{n+1}\sigma_\alpha&=\alpha^n m_{n+1}\sigma_\alpha+(s_n\sigma_\alpha^{-1}\triangleleft_{(1)} m_1)\sigma_\alpha\\&\quad+(s_n\sigma_\alpha^{-1})d_C\sigma_\alpha.\end{aligned}$$

By construction the terms we are calling $s_n\sigma_\alpha^{-1}$ are equal to $(\alpha^n-1)f_n$. That is, on homogeneous components of $C_n(V)$ of total homological degree $n+N$ (thus degree N in tensor powers of V) we have

$$s_n\sigma_\alpha^{-1}=\frac{1}{\alpha^N}s_n=(\alpha^n-1)\frac{1}{\alpha^{N+n}-\alpha^N}s_n=(\alpha^n-1)f_n.$$

Then our schematic equation becomes

$$(1-\alpha^n)(-m_1\triangleleft_{(1)} f_n+m_{n+1}+f_n\triangleleft_{(1)} m_1+f_nd_C)=0.$$

As long as $\alpha^n\ne 1$, we can divide by $1-\alpha^n$ and use Lemma 2.3 to get the schematic equation

$$m_1\triangleleft_{(1)} f_n^{-1}+m_{n+1}+f_n\triangleleft_{(1)} m_1+f_nd_C=0.$$

We claim that the left hand side of this equation is m'_{n+1}, which will complete the proof.

We return to the expressions (5) and (6) defining m'_{n+1}. There are three kinds of term in (5) which do not vanish from the conditions on m and f. One kind is the composition of m_1 with f_n^{-1} and f_0, one kind is m_{n+1}, and the third is the composition of f_n with m_1. For (6), the vanishing conditions on f and d_C imply that the only surviving term must be the composition of f_n with d_C Essentially by Lemma 1.5, we can write the overall calculation in our schematic pidgin as

$$m'_{n+1} = m_1 \triangleleft_{(1)} f_n^{-1} + m_{n+1} + f_n \triangleleft_{(1)} m_1 + f_n d_C$$

which is what we got from compatibility of m and s. □

Lemma 2.8. *The P_∞-automorphism S' has component $s'_0 : C_0(V) \to V$ equal to σ and s'_i vanishes on $C_i(V)$ in the range $1 \leq i \leq n+1$.*

Proof. The composition $s' : C(V) \to V$ takes the form

$$C(V) \to C(C(C(V))) \xrightarrow{f^{-1}} C(C(V)) \xrightarrow{s} C(V) \xrightarrow{f} V,$$

where the first map is induced by the decomposition of C. As in Lemma 2.7, the vanishing conditions for the degrees of C in which the maps f and f^{-1} are supported (0 or at least n) and the fact that $f_0 = f_0^{-1} = \mathrm{id}_V$ imply that $s'_i = s_i$ for $i < n$. For s'_n, we have the schematic equation

$$s'_n = f_n \triangleleft s_0 + s_n + s_0 \triangleleft f_n^{-1}$$

and so acting on the homogeneous component of $C_n(V)$ of total homological degree $n+N$, we have the equality

$$s'_n = \frac{\alpha^N}{\alpha^{N+n} - \alpha^N} s_n + s_n - \frac{\alpha^{N+n}}{\alpha^{N+n} - \alpha^N} s_n = 0,$$

as desired. □

This concludes the proof of Lemma 2.1. As the output of the lemma yields the input data with an increased index n, we can recursively arrive at the following.

$$(10)\quad (V, m^{[1]}, s^{[1]}) \xrightarrow{f^{[1,2]}} (V, m^{[2]}, s^{[2]}) \xrightarrow{f^{[2,3]}} \cdots \xrightarrow{f^{[n-1,n]}} (V, m^{[n]}, s^{[n]}) \xrightarrow{f^{[n,n+1]}} \cdots$$

where

1. the data $m^{[j]}$ is a P_∞ structure on V which vanishes on $C_i(V)$ for $i = 0$ and i in the range $2 \leq i < j+1$,
2. the data $s^{[j]}$ is a P_∞-automorphism of $(V, m^{[j]})$ which is equal to σ_α on V and vanishes on $C_i(V)$ in the range $1 \leq i < j$, and
3. the data $f^{[j,j+1]}$ is a P_∞-isomorphism from $(V, m^{[j]})$ to $(V, m^{[j+1]})$ intertwining $s^{[j]}$ and $s^{[j+1]}$ which is equal to the identity on V and vanishes on $C_i(V)$ for $i \notin \{0, j\}$.

We can continue this procedure up to the smallest n such that $\alpha^n - 1$ is not a unit. If $\alpha^n - 1$ is always a unit, then (10) is an infinite sequence.

Lemma 2.9. *If $\alpha^n - 1$ is a unit for all n, the transfinite composition*

$$f^{[1,\omega]} := \cdots \circ f^{[n,n+1]} \circ f^{[n-1,n]} \circ \cdots \circ f^{[1,2]}$$

is well-defined. In particular, the component $f_n^{[1,\omega]} : C_n(V) \to V$ is equal to the $C_n(V) \to V$ component of the finite composite

$$f^{[n,n+1]} \circ f^{[n-1,n]} \circ \cdots \circ f^{[1,2]}.$$

Proof. Because the 0th component of $f^{[m,m+1]}$ is the identity and all other components with index less than m are 0, the composition of $F^{[m,m+1]}$ with an arbitrary P_∞ map $G : W \to V$ has components $C_n(W) \to V$ equal to those of G for all indices n less than m. □

Remark 2.10. We were not able to find a comprehensible closed form expression for the coefficients involved in the transfinite composition, which seem to involve a summation over some special classes of decorated trees.

Now we are ready to prove the main theorem.

Proof of Main Theorem. By the homological perturbation lemma (Theorem 1.14) there is a P_∞-structure $m^{[1]}$ on $H_*(A,d)$, along with P_∞ quasi-inverses

$$\iota : (H_*(A,d), m^{[1]}) \rightleftarrows (A,d,m) : \pi.$$

Since these are quasi-inverses the composition $\pi\iota$ is homotopic to the identity of $H_*(A,d)$. But since this latter has no differential, this means $\pi\iota$ is equal to the identity. Define $s^{[1]}$ as the composition of the P_∞ morphisms π, $\hat{\sigma}$, and ι; then $s^{[1]}$ is automatically a P_∞ automorphism of $(H_*(A,d), m^{[1]})$. Its zeroth component is given by $\pi_0\hat{\sigma}\iota_0$. Since ι is a P_∞-morphism, in particular ι_0 lands in the cycles of A with respect to d, and then up to boundary terms, $\hat{\sigma}\iota_0 = \iota_0\sigma$. But then π_0 kills boundary terms so that

$$\pi_0\hat{\sigma}\iota_0 = \pi_0\iota_0\sigma = \sigma.$$

Then $m^{[1]}$ and $s^{[1]}$ are precisely the input data necessary for Lemma 2.1 and Lemma 2.9. If $\alpha^k -$ is a unit for all $k \le n$, then the finite composition

$$f^{[1,n]} := f^{[n-1,n]} \circ \cdots \circ f^{[1,2]}$$

is an isomorphism between $m^{[1]}$ and a P_∞-structure whose first n components vanish. If $\alpha^k - 1$ is always a unit, the transfinite composition $f^{[1,\omega]}$ constructed in Lemma 2.9 is an isomorphism between $m^{[1]}$ and the P_∞-structure m_*. □

Remark 2.11. This proof relies on the following consequences of the homological perturbation lemma for the base case:

(1) the chain complex $(H_*(A,d), 0)$ supports a P_∞-algebra structure $m^{[1]}$ with first component which is induced by m along with a quasi-isomorphism $(H_*(A,d), 0, m^{[1]}) \to (A,d,m)$, and
(2) the P_∞-algebra $(H_*(A,d), 0, m^{[1]})$ supports a P_∞-automorphism $s^{[1]}$ with first component σ (induced by $\hat{\sigma}$).

Any hypotheses guaranteeing these two conditions is sufficient for the argument.

3. A variant of the main theorem

We are interested in examples where the homology of the P-algebra A is concentrated in degree divisible by c for some integer c (an example is given by the cohomology of $\mathbb{CP}^n$ which is concentrated in even degrees). In such a situation we can "ignore" the zero cohomology groups and our main theorem becomes the following.

Theorem 3.1. *Let (A,d,m) be a differential graded P-algebra in R-modules such that the chain complex $(H_*(A,d), 0)$ can be written as a homotopy retract of (A,d). Assume further that the homology $H_*(A,d)$ is concentrated in degrees divisible by* *c. Let α be a unit in R and let $\hat{\sigma}$ be an endomorphism of (A, d, m) such that the induced map on $H_{cn}(A, d)$ is multiplication by α^n.*

- *If $\alpha^k - 1$ is a unit of R for $k \leq n$, then (A, d, m) is cn-formal as a P-algebra.*
- *If $\alpha^k - 1$ is a unit of R for all k, then (A, d, m) is formal as a P-algebra.*

Observe that if α has a c-th root in our ring R, then this theorem is our main theorem with α replaced by $(\alpha)^{1/c}$. In general this variant is proved by adapting the proof of Lemma 2.1 in an obvious manner.

4. Applications

4.1. **Hodge theory.** Let us reinterpret the classical Deligne–Griffiths–Morgan–Sullivan [DGMS75] result in the light of the present paper. This has already been done indirectly by Sullivan [Sul77, §12]. There he argues that complex formality of Kähler manifolds (proven by other means in [DGMS75]) implies that degree twisting automorphisms lift to the chain level. Then rational degree twisting automorphisms lift to the chain level, and therefore the rational homotopy type of a Kähler manifold is formal.

We outline a more direct proof along the same lines, sketching a way to obtain real-valued chain level lifts of the degree twisting automorphism. Arguably this is not the most natural way to prove such results so we do not provide full details.

Let X be an algebraic variety over $\mathbb{C}$. Then the cohomology of $\mathbb{C}$ has the structure of a mixed Hodge structure. If X is smooth and proper, then the n-th cohomology group is a pure Hodge structure of weight n but in general a given cohomology group can involve several different weights. In [CH17], Cirici and the second author explained how one can view a model $A^*(X)$ for the singular cochains of X as a commutative algebra in the ∞-category of chain complexes of mixed Hodge structures. The abelian category of real mixed Hodge structure is a Tannakian category. Denote its Tannakian Galois group by $\mathrm{G}_{\mathrm{MHS}}$ and denote by $\mathrm{G}_{\mathrm{PHS}}$ the Tannakian Galois group of pure Hodge structures. There is a split surjection $p : \mathrm{G}_{\mathrm{MHS}} \to \mathrm{G}_{\mathrm{PHS}}$ that is Tannaka dual to the inclusion of pure Hodge structures into mixed Hodge structures. The splitting $s : \mathrm{G}_{\mathrm{PHS}} \to \mathrm{G}_{\mathrm{MHS}}$ is Tannaka dual to the map that sends a mixed Hodge structure to the associated graded of its weight filtration. See [Gon11] for more details about this. In particular, using some non-trivial homotopical machinery, it is possible to construct an action of $\mathrm{G}_{\mathrm{MHS}}(\mathbb{R})$ on a model $A^*(X)$ of the real homotopy type of X. We can restrict this action along s and we get an action of $\mathbb{C}^\times = \mathrm{G}_{\mathrm{PHS}}(\mathbb{R})$ on $A^*(X)$. We have the following fact about this action.

Proposition 4.2. *If $H^k(X, \mathbb{R})$ is a cohomology group of X whose mixed Hodge structure is pure of weight n, then the action of $x \in \mathbb{R}^\times \subset \mathbb{C}^\times$ on $H^k(X)$ is given by multiplication by $|x|^{2n}$.*

Proof. This can be found in [DMOS82, Paragraph 2.31, p.145]. □

Hence we can prove the following theorem.

Theorem 4.3. *Let X be a smooth projective complex variety, or more generally a variety satisfying the property that $H^k(X, \mathbb{R})$ is a pure Hodge structure of weight k for all k. Then there exists a model $A^*(X)$ for the real homotopy type of X that is formal.*

4.4. **Formality of the little disks operad.** Following Petersen, we can prove formality of the little disks operad with rational coefficients by using the fact that, for any α in $\mathbb{Q}^\times$, there exists an automorphism of $C_*(\mathcal{D}_2, \mathbb{Q})$ that induces the grading automorphism σ_α on the homology (see the proof of the main Theorem of [Pet14]). Our Main Theorem gives an alternative proof of the Proposition in [Pet14]. Note that this proof is closer in spirit to the intuition developed in the last section of [Pet14].

4.5. **Complement of subspace arrangements.** In this subsection, we denote by K a finite extension of $\mathbb{Q}_p$. The residue field of the ring of integers of K is isomorphic to $\mathbb{F}_q$ for q some power of p. We denote by ℓ a prime number different from p and we denote by h the order of q in $\mathbb{F}_\ell^\times$.

Let X be a complement of a hyperplane arrangement in a complex vector space. We say that X is defined over K if there exists a hyperplane arrangement in a K-vector space that becomes isomorphic to the one defining X after extending the scalars along an embedding $\iota : K \to \mathbb{C}$.

Theorem 4.6. *Let X be a complement of a hyperplane arrangement. Assume that X is defined over K. Then the dg-algebra $C^*(X_{an}, \mathbb{Z}_\ell)$ is $(h-1)$-formal*

Proof. Indeed in that case, by standard comparison results in étale cohomology, we have a quasi-isomorphism of dg-algebras

$$C^*(X_{an}, \mathbb{Z}_\ell) \simeq C^*_{ét}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$$

where $\mathcal{X}$ is the complement of a hyperplane arrangement defined over K that exists by assumption. Let σ be a Frobenius lift, i.e. an element of $\mathrm{Gal}(\overline{K}/K)$ that maps to a generator of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Then the action of σ on $H^n_{ét}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is given by multiplication by q^n (this is proved in [CH18, Lemma 8.10] with $\mathbb{Q}_\ell$ coefficients but this is enough since these cohomology groups are torsion free). We are thus precisely in the situation of our main theorem. □

Remark 4.7. This theorem is the codimension 1 version of [CH18, Theorem 8.11]. Note that, in contrast to [CH18], we do not make the assumption that X_{an} is simply connected. The higher codimension version of [CH18, Theorem 8.11] can also be proved using the methods of the current paper. It should also be noted that we obtain a formality result over $\mathbb{Z}_\ell$ and not just over $\mathbb{F}_\ell$ as in [CH18].

Remark 4.8. Note that the condition of being defined over K cannot be dropped. Indeed for each prime ℓ, Matei in [Mat06], gives an example of a hyperplane arrangements in $\mathbb{C}^3$ whose complement has a non-trivial Massey products in $H^2(-, \mathbb{F}_\ell)$. However, the equations of his hyperplanes involve ℓ-th roots of unity. If a p-adic field K has ℓ-th roots of unity then the residue field $\mathbb{F}_q$ must have ℓ-th roots of unity as well and this implies that ℓ divides $q-1$. But in that case q is congruent to 1 modulo ℓ and therefore the previous theorem is an empty statement. This is in sharp contrast with the case of rational coefficients where all complements of hyperplane arrangements are formal.

Remark 4.9. On the other hand, if the hyperplane arrangement is defined over $\mathbb{Q}$, then it is defined over $\mathbb{Q}_p$ for every p. We are then free to pick p such that p is of order $(\ell-1)$ in $\mathbb{F}_\ell^\times$ and we obtain $(\ell-2)$-formality for such arrangements. Example of such arrangements are the A_n, B_n, C_n and D_n arrangements.

We also have a similar result for complements of toric arrangements. We first recall the relevant definition. A *character* of $(\mathbb{C}^*)^d$ is an algebraic group homomorphism $(\mathbb{C}^*)^d \to \mathbb{C}^*$. It is straightforward to check that any character is of the form

$$(z_1, \ldots, z_d) \mapsto z_1^{n_1} \ldots z_d^{n_d}$$

with $n_1, \ldots, n_d$ a sequence of integers. Given a character of $(\mathbb{C}^*)^d$ and a non-zero complex number a, we denote by $H_{\chi,a}$ the subvariety of $(\mathbb{C}^*)^d$ defined by the equation

$$\chi(z_1, \ldots, z_d) = a$$

Definition 4.10. A complement of a toric arrangement is an open subspace of $(\mathbb{C}^*)^d$ of the form

$$(\mathbb{C}^*)^d - \bigcup_{i=1}^{n} H_{\chi_i, a_i}$$

where each χ_i is a character and each a_i is a non-zero complex number.

We say that a complement of a toric arrangement X is defined over K if there exists an embedding $\iota : K \to \mathbb{C}$ such that the coefficient a in the equation of each $H_{\chi,a}$ is in the image of ι. Given such a choice of ι we can construct a variety $\mathcal{X}$ over K given as the open complement in $\mathbb{G}_m^d$ of the closed subsets $H_{\chi,a}$ (or more precisely $H_{\chi,\tilde{a}}$ where $\tilde{a}$ is the preimage of a) and we have

$$X = \mathcal{X} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathbb{C})$$

Proposition 4.11. *Let X be a complement of a toric arrangement that is defined over K. Then the dg-algebra $C^*(X, \mathbb{Z}_\ell)$ is $(h-1)$-formal.*

Proof. Again, by comparison between étale and singular cohomology, it suffices to prove that $C^*_{ét}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is formal. We claim that for any choice of Frobenius lift σ in $\mathrm{Gal}(\overline{K}/K)$ the action of σ on $H^n_{ét}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is given by multiplication by q^n. Indeed, by [dD15] the cohomology of the complement of a toric arrangement is torsion free, it follows that it is enough to prove that σ acts on $H^n_{ét}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$ by multiplication by q^n. The proof is then completely analogous to the proof of [Dup15, Theorem 3.8] but using étale cohomology instead of de Rham cohomology. We can therefore conclude as in the proof of Theorem 4.6. □

Remark 4.12. This result about formality of complement of hyperplane arrangements and toric arrangements with coefficients in $\mathbb{Z}_\ell$ is new as far as the authors know.

Using étale cohomology with coefficients in $\mathbb{Q}_\ell$ instead of $\mathbb{Z}_\ell$, we can also prove formality (without bound) of cohomology of complements of hyperplane and toric arrangements with $\mathbb{Q}_\ell$-coefficients. This gives an alternative proof of the results of Brieskorn and Dupont (proved respectively in [Bri73] and [Dup15]). Let us mention however that the étale cohomology method only yields formality if the arrangement is defined over a p-adic field. The results we get are thus less general than those of Brieskorn and Dupont.

4.13. **Coformality of configuration spaces.** Recall that, by the work of Quillen, the homotopy type of simply connected rational spaces is captured by a differential graded Lie algebra. One says that a space is coformal if this Lie algebra is formal. By [Sal17, Corollary 1.2], asking for a space X to be coformal is equivalent to asking for the dg-algebra $C_*(\Omega X, \mathbb{Q})$ to be formal (by ΩX we mean a stricly associative model for the loop space of X). The advantage of this second definition is that it can be generalized to coefficient rings that are not $\mathbb{Q}$-algebras.

We are interested in the coformality of the configuration space of n distinct ordered points in $\mathbb{R}^d$ denoted $\mathrm{Conf}_n(\mathbb{R}^d)$. These spaces are known to be rationally coformal. We have the following theorem about coformality with $\mathbb{Z}_p$-coefficients.

Theorem 4.14. *Let $d \geq 3$. Let $X = \mathrm{Conf}_n(\mathbb{R}^d)$. Let p be a prime number. The dg-algebra $C_*(\Omega X, \mathbb{Z}_p)$ is $(p-2)(d-2)$-formal.*

Proof. We can first replace X by its p-completion which we will do implicitly from now on. In [BdBH19], an action of GT_p, the p-complete Grothendieck–Teichmüller group is constructed on X. As a consequence of this action, it is shown in [BdBH19, Proposition 8.2] that for any unit α in $\mathbb{Z}_p$, there exists an automorphism $\alpha^\sharp$ of X that acts by multiplication by α in homological degree $(d-1)$. Since this is the bottom non-vanishing homology group of X, using the Hurewicz isomorphism twice, we can identify this group with $H_{d-2}(\Omega X, \mathbb{Z}_p)$. By [CG02, Theorem 2.3], we know that the homology of ΩX is torsion-free, concentrated in degree divisible by $(d-2)$ and is generated as an algebra by classes of degree $(d-2)$. Since the action of $\alpha^\sharp$ exists at the space level, it is compatible with the dg-algebra structure on $C_*(\Omega X, \mathbb{Z}_p)$ and we deduce that $\alpha^\sharp$ acts as multiplication by α^k in homological degree $k(d-2)$. We can pick an α whose residue modulo p is a generator of the group of units of $\mathbb{F}_p$. For such an α, the number $\alpha^k - 1$ is a unit in $\mathbb{Z}_p$ for $k \leq p-2$. We are thus precisely in the situation of Theorem 3.1. □

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